Competitive intensities and their effects on firm performance

Joao Montez
London Business School

Francisco Ruiz-Aliseda
Ecole Polytechnique

Michael D. Ryall
University of Toronto

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1 Introduction

A central idea underlying the traditional strategy literature on firm performance is that profitability is inversely proportional to competitive intensity. For example, the literature on firm positioning (e.g., Caves and Porter, 1977; Porter 1979) examines the role of industry mobility barriers in suppressing competition which would, otherwise, erode positive economic profit. The “resource-based view” (e.g., Wernerfelt, 1984; Barney, 1991) presents a similar logic, under which resource mobility barriers support the consistent attainment of economic profit.\(^1\) So deeply embedded is this traditional line of thinking that we imagine an assertion to the contrary might well be considered controversial by many readers.

Our paper challenges the conventional understanding of the relationship between competition and performance. Our main contribution is to formalize a novel notion of competitive intensity and use it to establish new results on the relationship between competition and performance. We show that, consistent with conventional wisdom, the intensity of competition for a firm’s transaction partners does, indeed, reduce its ability to appropriate value from them. However, we also show that the intensity of competition for a firm increases the amount its transaction partners must pay to induce it to deal with them. Our results can be used to determine whether these positive and negative effects are, on balance, favorable or unfavorable to a firm given its competitive setting.

These findings add to a growing stream of formal, theoretical work in strategy based upon cooperative game theory (e.g., Brandenburger and Stuart 1996, MacDonald and Ryall 2004).\(^2\) A cooperative game begins with a set of agents who are free to engage one another in productive transactions, taking as primitive the amounts of economic value that the various combinations of agents could create should they agree to transact with one another. By imposing two competitive consistency conditions (known as the “core” conditions), results are derived linking the agents’ productive opportunities to the amounts they appropriate.

There are two essential features of competition under this framework. The first is that competition is inherently neutral in its effect – that is, it can work to the advantage or disadvantage of

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\(^1\)Here, we cite a few seminal papers to illustrate our point – which is in no way intended to minimize the richness or depth of the vast literature on this topic. This logic is also apparent in transaction cost economics (Williamson, 1971), which considers switching barriers between governance forms, as well as in foundational contributions like Nelson and Winter (1982), Porter (1996), Rumelt (1977), Teece et al. (1977), Eisenhardt and Martin (2000), Rivkin (2000), and Foss and Knudsen (2003).

\(^2\)This literature, and the relationship of our paper to it, is more carefully discussed in the next section.
a firm, depending upon the specifics of its situation.\(^3\) The second is that, in general, competition only imposes loose bounds on a firm’s performance. That is, there is typically a *range* of profits that meet the consistency conditions implied by competition (known as “core intervals”). Strategy scholars are interested in how this alternative conception of competition informs our understanding of strategy’s existing catalog of theoretical ideas. How does competitive intensity affect the performance bounds on a market’s participants? How can the insights of this approach be applied to real world settings? What, if anything, does variation in competitive intensity imply about a firm’s selection of strategy? Which of a firm’s resources determine the lower bound on its performance due to competition and which the point within its range due to super-competitive factors (e.g., bargaining skill)? This paper provides some answers to the first two of these questions.

In addition to our main results on competitive intensity, we make several complementary contributions. Although cooperative game theory is useful in deriving general insights into the relationship between competition and firm performance, it also presents obstacles to empiricists and business practitioners who wish to test and, respectively, use them under real-world conditions. The problem is two-fold. First, what an agent appropriates is typically shaped by a constellation of transactions that *could have been* consummated (but were not). Thus, many of the values required for a complete analysis are counterfactuals. Second, even if one were able to construct the counterfactual values based upon, say, estimation of underlying primitives, the number of values required is exponentially increasing in the number of market participants. The result is potentially insurmountable computational issues.

These issues are not isolated to strategy applications. Thus, a stream of “core catcher” work in economics has as its goal the estimation of competitive ranges of appropriation (core intervals) from subsets of relevant values. Our notion of competitive intensity is a new type of core catcher that, in our judgment, is likely to be useful in real-world strategy applications.

Finally, in order to illustrate the insights associated with our general results, we develop a novel differentiated products model that admits horizontal and vertical differentiation as well as multilateral deals (e.g., supplier-firm-distributor-buyer chains) and the inclusion of capacity constraints (c.f. Kreps and Scheinkman 1983). We show how differences in competitive intensity can result in a firm that is vertically superior to a rival being outperformed by that rival. By endogenizing capacity choices before firms compete à la Brandenburger and Stuart’s (2007) bi-form game setup, we

\(^3\)The importance of this insight for strategy is explored in Gans, Macdonald, and Ryall (2008).
show that the market never exhibits firms with idle capacity (under very weak conditions). Firms refrain from expanding capacity because doing so lessens competitive intensity for them while, at the same time, increasing competitive intensity for their buyers. The overall effect is a strict shift of their competitive range in the wrong direction.

Reading the management scholarship as well as the popular business press, it is not uncommon to run across normative admonitions that managers should strive to maximize the value created by their firms. Although sophisticated readers well understand the distinction between value creation and value appropriation, intuition suggests the two should be positively related. However, we show in our differentiated products model that, sometimes, an firm’s successful attempt to “create more value” actually harms its profitability. For instance, capacity addition by a firm is equivalent to the firm lowering the cost of buyers who were not served previously. Although this may allow the firm to steal buyers away from a rival, it may also leave the firm in a weak competitive position vis-à-vis its buyers. The firm’s new buyers become targets for the rival (who wishes to fill excess capacity created when the firm lured its buyers away). Our results on competitive intensity provide a nice insight to this outcome. By bringing buyers into its transaction network, competitive intensity for the firm declines, thereby lowering its minimum. By creating excess capacity in a rival, competitive intensity for its buyers increases, thereby also lowering its maximum. The overall effect is unambiguously bad for the firm (even though it creates more value with the addition of new buyers).

The paper proceeds as follows. In Section 2, we discuss the related literature. In Section 3, we set up a general cooperative game, define our notions of competitive intensity and XX, and use them to draw conclusions about a firm’s appropriation possibilities. In Section 4, we introduce our differentiated products model and use it to establish sharp results on the effects of competition on performance in a familiar context. Section 4 presents our concluding thoughts. Proofs are relegated to the Appendix.

2 Related literature

[Forthcoming]

4This finding relates to the seminal contribution of Kreps and Scheinkman (1983), which provides a microfoundation for Cournot (quantity) competition based on simultaneous capacity choices followed by capacity-constrained Bertrand (price) competition.

5This positive relationship is strongly implied in the famous paper by Brandenburger and Stuart (1996) on “added-value” strategies.
3 General results

We adopt the following notational conventions. Sets are indicated with capital letters. Elements of sets, scalars, vectors and functions are all represented by small letters. Terms defined in the body text are italicized at the point of definition.

3.1 Preliminaries

Value Creation We consider a market situation with a set of participants $N \equiv \{1, \ldots, n\}$. We will call each participant an agent. A characteristic function $v : 2^N \rightarrow R_+$ represents the value-producing opportunities available to the agents in $N$ with $v(\emptyset) = 0$. $v(G)$ is the aggregate value that could be produced by a subset of agents $G \subseteq N$. The elements $(N, v)$ define a cooperative game associated with this market.

A characteristic function $v$ is superadditive if disjoint groups of agents do not produce less value if they can coordinate their activities than if they cannot—or formally if $v(G \cup S) \geq v(G) + v(S)$ for all $G, S \subseteq N$ such that $G \cap S = \emptyset$. Naturally $v(N)$ is the aggregate value we expect to be produced.

Value Appropriation How and to what extent does competition determine a firm’s ability to appropriate value? The essential feature of competition in a market is that agents are free to seek out the transactions that maximize the value they appropriate. Define a distribution of value as $\pi \equiv (\pi_1, \ldots, \pi_n) \in R^n$, where $\pi_i$ is a scalar indicating the amount of value appropriated by agent $i \in N$.

Each agent $i$ wishes to maximize his or her own appropriation level $\pi_i$. Without attempting to model all the intricacies of the deal-making process, agent $i$’s freedom to negotiate the most attractive transactions suggests two broad restrictions on this appropriation. The first, and perhaps most obvious, restriction is that the agents in an industry can never appropriate more value in aggregate than the amount they actually create. Formally,

$$\sum_{i \in N} \pi_i \leq v(N).$$

(1)

Second, given the set of deals that give rise to $v(N)$ and a distribution $\pi$, it cannot be in a situation of complete information that some agent $i$ is a member of a subset of agents $G$ that could engage
in some other transactions that would make them all better off—else, agent \( i \) would be able to convince those agents to make those alternative transactions by promising them a larger payoff and keep the remainder to himself. Such objections to a particular payoff \( \pi \) lead to the following notion of stability:

\[
\sum_{i \in G} \pi_i \geq v(G), \text{ for all } G \subseteq N. \tag{2}
\]

A distribution of value is said to be consistent with competition when (1) and (2) are satisfied. The set of all such \( \pi \) is commonly referred to as the core.

An implicit assumption in the notion of stability, i.e. in (2), is that all agents know the payoffs the remaining agents are receiving. This information is essential if a player is to detect profitable defections with some subset \( G \).

**Competitive intervals** The interpretation of Conditions (1) and (2) is that agents in a market aggressively compete to engage in the deals required in the production of \( v(N) \). Typically, a tension arises between the potential value available and the value that must be spread around via \( \pi \) in order to dissuade some agents from engaging in some other of the alternatives summarized by \( v \).

This tension increases as the values \( v(G) \) of different subsets \( G \subseteq N \) increase relative to \( v(N) \)—meaning, the ways in which (1) can be used to satisfy (2) become more restrictive. This tension is, in essence, what is meant by “competitive intensity.”

For each agent \( i \in N \), those payoffs that are consistent with competition form what is called a competitive interval, denoted \([\pi_i^{\text{min}}, \pi_i^{\text{max}}]\). Thus \( \pi_i^{\text{min}} \) is the payoff guaranteed by competition alone. The greater the competitive intensity the narrower are the agents’ competitive intervals. Competition fully determines an agent’s payoff if \( \pi_i^{\text{min}} = \pi_i^{\text{max}} \). In general \( \pi_i^{\text{min}} \leq \pi_i^{\text{max}} \) and whatever \( i \) is able to appropriate above \( \pi_i^{\text{min}} \) is due to factors other than competition (persuasive skills, bargaining advantages, market norms, etc.).

**Added Value** A well-known notion of individual productivity is an agent’s added value. Define \( i \)’s added value to \( G \) as \( av_i(G) \equiv v(G \cup i) - v(G \setminus i) \), where \( G \setminus i \) indicates \( G \) with \( i \) removed from \( G \) and \( G \cup i \) indicates \( G \) with \( i \) added. \( av_i \) is nonnegative if \( v \) is superadditive. In a similar way we can define the value added of a group of agents \( G \) to the group \( S \) by \( av_G(S) = v(S \cup G) - v(S \setminus G) \). The superadditivity of \( v \) implies that \( av_G(S) \) is nonnegative for all \( G, S \subseteq N \); in particular, \( av_i(S) \geq 0 \).
for all \( S \subseteq N \) and all \( i \in N \).

We thus have that \( av_i(N) = v(N) - v(N \setminus i) \) represents agent \( i \)'s added value to the industry as a whole and \( av_G(N) = v(N) - v(N \setminus G) \) the added value of a group of agents \( G \) to the industry as a whole.

A natural and well known consequence of competitive consistency requirements (1) and (2) is that a division of the surplus \( \pi \) is consistent with competition if and only if

\[
v(G) \leq \sum_{i \in G} \pi_i \leq av_G(N) \text{ for all } G \subseteq N.
\]

It is intuitive that any group of agents cannot get more than they jointly add to the economy since otherwise the remaining agents would find it profitable to separate from \( N \) and share among themselves the value they are able to produce on their own.

In particular this means that in any such \( \pi \) agent \( i \) may receive at most his marginal contribution and at least his stand alone payoff, i.e., \( v(i) \leq \pi_i \leq av_i(N) \).

**Value Partitions** We refer to a collection of \( N \), denoted \( \mathcal{P} \), as a value partition if: (i) \( \mathcal{P} \) is a partition of \( N \), i.e., division of \( N \) into non-overlapping subsets, (ii) \( v(N) = \sum_{G \in \mathcal{P}} v(G) \) and iii) for each \( G \in \mathcal{P} \) there are no \( S, T \subset G \) with \( S \cap T = \emptyset \) such that \( v(G) = v(S) + v(T) \). A value partition is non-trivial if there is more than one productive element. For a given \( (N, v) \), there may be multiple value partitions, i.e., different configurations of agents that lead to the creation of \( v(N) \). We denote the set of all value partitions by \( \mathcal{P} \).

We refer to each element in a value partition as value networks and, henceforth, use \( X \) to denote such a group (as distinguished from an arbitrary group \( G \)). For example, in the context of an industry, distinct supply chains running from raw materials to buyers are value networks.

For any value partition \( \mathcal{P} \) and distribution of value \( \pi \) consistent with competition we must have that

\[
\sum_{i \in X} \pi_i = v(X) \text{ for all } X \in \mathcal{P}.
\]

That is, the agents of a value network appropriate among themselves precisely the value produced via their joint economic activities. If \( \mathcal{P} \) represents a set of distinct industry supply chains, then this says that all the value produced by a chain—aggregate buyer utility minus the economic costs of production—is distributed only among the members of that chain. Notice as well that, since
\( \pi_i \leq av_i(N) \), this implies that

\[
v(X) = \sum_{i \in X} \pi_i \leq \sum_{i \in X} av_i(N) \text{ for all } X \in \mathcal{P}.
\]  

(5)

Note that for a given \((N,v)\) there may be multiple value partitions. In the real-world we should observe one of those value partition by observing the agents’ joint productive activities: agents linked to one another directly or indirectly via their contemporaneous transactions would identify a value network. Together, these networks would constitute the de facto value partition for that market.

In order to emphasize this interpretation, imagine that one partition, denoted \(\mathcal{P}^*\), is the one implied by the actual transactions that give rise to \(v(N)\) in the market represented by \((N,v)\). We refer to \(\mathcal{P}^*\) as the focal value partition, its elements \(X^*\) the focal value networks and denote by \(X^*_i\) the focal value network of which agent \(i\) is a member.

### 3.2 Competitive intensity

Suppose each agent \(i\) understands the value creation opportunities. In an actual market, computing the exact bounds on an agent’s competitive interval requires assessing the value-creation possibilities of \(2^n - 1\) subsets of \(N\) and then solving the linear optimization programs associated with \(\pi_i^{\min}\) and \(\pi_i^{\max}\), each involving \(2^n - 1\) constraints. This becomes infeasible at even moderate levels of \(n\).

Suppose in addition that agents cannot observe the exact payoffs of the remaining players when it comes to contemplating deviations. In this case it becomes harder to justify objections behind the notion of stability in (2). What reasonable objections could an individual player \(i\) still make to try to improve her payoff based on her limited information about others’ payoffs? What is the minimum payoff \(i\) can get based on such objections? Will more primitive objections still allow a player to exactly predict her competitive interval?

We argue that management can still estimate a reasonable lower bound on their minimum level of appropriation by: 1) identifying the alternative supply chain to which they could add some value; and 2) estimating what that added value would be. To the extent they can repeat this exercise for their trading partners, they will also determine an upper bound on what they can appropriate.
3.2.1 Weak competitive intensity

Consider an alternative value network in the focal value partition $X \in \mathcal{P}^*$ such that $i \notin X$. If $av_i(X) > 0$ then the agents in $X$ would have created additional value for $X$ had they convinced $i$ to join them rather than $X_i^*$. In this sense, we say that $X$ competes with $X_i^*$ for $i$. Referring to the set $\mathcal{P}^* \setminus X_i^*$ as the competitive periphery of agent $i$, we have the following notion.

**Definition 1.** The weak competitive intensity for $i \in N$ in a focal value partition $\mathcal{P}^*$ is:

$$w_i^* \equiv av_i(C_i^*), \text{ where } C_i^* \equiv \arg \max_{X \in \mathcal{P}^* \setminus X_i^*} av_i(X)$$

That is, $C_i^* \in \mathcal{P}^*$ is any value network, to which $i$ does not belong, where $i$ would be the most productive (by superadditivity, $av_i(X) \geq v(i)$ for all $X \in \mathcal{P}^* \setminus X_i^*$). For example, buyers in a firm’s competitive periphery are those who would have preferred to trade with that firm but could not (say, as a result of capacity constraints). So any value network $C_i^*$ provides $i$ with a potential external alternative to $X_i^*$. This has implications for $i$’s appropriation because it affords her a credible threat to abandon her own focal value network $X_i^*$ and to join $C_i^*$, and $C_i^*$ should be willing to bid up to $w_i^*$ for $i$.

While the competitive intensity for $i$ protects the share she can get of the surplus, the competitive intensity for the remaining agents in $X_i^*$ will also limit what $i$ can expect to get. The following definition captures this side of competition.

**Definition 2.** The weak competitive intensity for $i$’s partners is

$$\sum_{j \in X_i^* \setminus i} w_j^*,$$

and the weak residual of $i$ under $\mathcal{P}^*$ is what is left from $v(X_i^*)$ after paying $i$’s partners their weak competitive intensity but no more than $i$’s added value to $X_i^*$, i.e.,

$$w_{-i}^* \equiv \min \left\{ v(X_i^*) - \sum_{j \in X_i^* \setminus i} w_j^*, av_i(X_i^*) \right\}. \quad (7)$$

8
Intuitively, intensity of competition for $i$’s partners places an upper bound on $i$’s appropriation of the share of the value created within $i$’s own value network, which is complemented by the internal threat of those partners to exclude $i$ from $X^*_i$.

Competitive intensity gives a safe lower bound on what an agent can expect to get due to competition, while the intensity of competition for her trading partners provides an optimistic bound on what that agent could get. We have:

**Proposition 1.** Given a focal value partition $\mathcal{P}^*$, for all $i \in N$,

$$\pi^\text{min}_i \geq w^*_i,$$

$$\pi^\text{max}_i \leq w^*_{-i}. \quad (8)$$

**Proof.** We first show that $\pi^\text{min}_i \geq w^*_i = \max_{X \in \mathcal{P}^* \setminus X^*_i} \text{av}_i(X)$. If this were not the case, $i$ could always join any $Y \in \arg \max_{X \in \mathcal{P}^* \setminus X^*_i} \text{av}_i(X)$ (or refuse joining any value network if we do not have $w^*_i > v(i)$). Since we know that $v(Y) = \sum_{j \in Y} \pi_j$, a new value network formed by $i$ and the members of $Y$ would make all of them better off because of the positive value added by $i$, which would contradict the hypothesis that $\pi^\text{min}_i$ is a possible core allocation.

Since it clearly holds that $\pi^\text{max}_i \leq \text{av}_i(X^*_i)$ (otherwise, the other members of $X^*_i$ would improve by excluding $i$), we conclude the proof by demonstrating that $\pi^\text{max}_i \leq v(X^*_i) - \sum_{j \in X^*_i \setminus i} w^*_j$. To show this, note if $i \in X^*_i$ receives the maximal payoff she can get in a core allocation that we must still have that any other player $j \neq i$ in $X^*_i$ gets a payoff $\pi_j$ at least as large as $\pi^\text{min}_j$. Because $v(X^*_i) = \pi^\text{max}_i + \sum_{j \in X^*_i \setminus i} \pi_j$, we must then have that $v(X^*_i) \geq \pi^\text{max}_i + \sum_{j \in X^*_i \setminus i} \pi^\text{min}_j$, so the result that $\pi^\text{min}_j \geq w^*_j$ for all $j \in X^*_i \setminus i$ yields that $\pi^\text{max}_i + \sum_{j \in X^*_i \setminus i} \pi^\text{min}_j \geq \pi^\text{max}_i + \sum_{j \in X^*_i \setminus i} w^*_j$. We therefore have $v(X^*_i) \geq \pi^\text{max}_i + \sum_{j \in X^*_i \setminus i} w^*_j$, which ends the proof. \qed

This notion of competitive intensity is only based on the value added by $i$ to his competitive periphery—that is, the agents with whom it does not transact under the focal value partition but with whom productive transactions are available. The agents in $i$’s value network must compete with those in $i$’s competitive periphery by offering $i$ a sufficient share of value that keeps $i$ engaged in their network. Of course, $i$ must compete in similar fashion for the engagement of its network trading partners.

It may seem surprising that the lower bound of what $i$ may get does not depend on the value added by $i$ to his own value network $\text{av}_i(X^*_i)$. Moreover the upper bound will also not depend on
i’s added value whenever the competitive intensity for his partners is sufficiently high relative to his added value.

As we shall see, this is in fact a general feature, i.e. the value added of a player to his trading partners is not in general an essential determinant of what a player may get in a competitive situation. We will also show that players with a larger added value do not unnecessarily receive higher payoffs than those with lower added values, that an increase in a player’s added value may not increase the minimum nor the maximum he receives—an increase in a player’s added value may in fact decrease both payoffs. The importance and main contribution of the notion of competitive periphery is to give us a language to intuitively explain these results.

The notion of competitive intensity can be strengthened in a obvious way by considering not only the focal value partition $\mathcal{P}^*$ but all value partitions that could have existed, i.e. all $\mathcal{P} \in \mathcal{P}$. Because Proposition 1 holds for any arbitrary (“focal” ) value partition, it holds for all value partitions. Therefore, $\pi_i^{\text{min}}$ must be at least as large as i’s added value to an outside value network under the value partition that maximizes that amount (i.e., even if the maximizing partition is not the “actual” one in which i finds itself).

We can therefore replace in expression (6) the set over which the maximization takes place by

$$X \in \mathcal{P} \in \mathcal{P}|i \notin X.$$ 

If we do this for all agents, and follow the approach outlined above, since the maximization takes arguments from a larger set we will find tighter bounds. The example below shows the usefulness of this additional step.

**Example 1.** Consider the market for a homogeneous good sold by two firms labeled $a$ and $b$. Each of them holds two units of production capacity and can produce any of the two units at no cost. There also exist three identical buyers labeled 1, 2 and 3. If each buyer has unit demand and values the good at $u > 0$. It can be shown that the core of this game is a singleton, with $\pi_a = \pi_b = 0$ and $\pi_i = u$ for all $i \in \{1, 2, 3\}$. To apply Proposition 1, suppose the focal value partition is $\mathcal{P}^* = \{\{a, 1, 2\}, \{b, 3\}\}$. (the set $\mathcal{P}$ has five extra non-trivial value partitions). Then $v(\{a, 1, 2\}) = 2u$ and $v(\{b, 3\}) = u$ and $w_1^* = w_2^* = u$ since either of buyers 1 and 2 bring an added value of $u$ if they individually join the value network in which firm $b$ has idle capacity. Also, $w_3^* = 0$ (since firm $a$ is producing at full capacity) and $w_a^* = w_b^* = 0$ (since products are homogeneous). As a result, we have that
\[ \pi_a^{\text{max}} \leq 0 \leq \pi_a^{\text{min}} \text{ and } \pi_i^{\text{max}} \leq u \leq \pi_i^{\text{min}} \text{ for } i \in \{1, 2\}, \text{ so } \pi_a = 0 \text{ and } \pi_1 = \pi_2 = u. \]  
In addition, we have that \(0 \leq \pi_b^{\text{min}} \leq \pi_b^{\text{max}} \leq u\) and \(0 \leq \pi_3^{\text{min}} \leq \pi_3^{\text{max}} \leq u\). Not all agents’ competitive intervals are precisely pinned down, although some of them are. Take now an alternative \(P = \{(b, 2, 3), (a, 1)\}\), which also belongs to \(P\). We can then see that \(\pi_b^{\text{max}} \leq 0 \leq \pi_b^{\text{min}} \leq u\) and \(\pi_3^{\text{max}} \leq \pi_3^{\text{min}} \leq u\), so it holds that \(\pi_b = 0\) and \(\pi_3 = u\). Coupled with our previous findings that \(\pi_a = 0\) and \(\pi_1 = \pi_2 = u\) in the focal value partition \(P^*\), the notion of competitive intensity allows us to directly characterize the unique distribution of value consistent with competition in a very simple manner.

Sometimes, though, there are no other value partitions than the focal one. Is there another way to extend the notion of competitive periphery and to sharpen the predicted bounds?

### 3.2.2 Strong competitive intensity

To motivate the next concept, we continue with Example 1. Under the focal value partition \(P^* = \{(a, 1, 2), (b, 3)\}\), the notion of weak competitive intensity pinned down sharp bounds for some but not all players. For example, \(w_3^* = 0\) does not exactly pin down \(\pi_3^{\text{min}}\) because buyer 3 does have a credible threat against firm \(b\) that is not captured by \(w_3^*\). Such a threat is based on buyers 1 and 2 leaving a low rent to firm \(a\) based on the strong competition of firm \(b\) for any of them. The high competitive intensities of buyers 1 and 2 results in a low payoff for firm \(a\), which leaves the door open for a profitable deal between firm \(a\) and buyer 3 if the latter is getting a low payoff too. Thus, firm \(a\) is getting at most \(u - w_i^*\) of its transaction with buyer \(i \in \{1, 2\}\), so buyer 3 could offer herself to replace buyer \(i \in \{1, 2\}\) and compensate firm \(a\) with \(u - w_i^*\). Doing so would guarantee a payoff to buyer 3 equal to \(\max_{i=1,2} \{u - (u - w_i^*)\}\) (since buyer 3 would target the least profitable buyer for firm \(a\)). We can thus strengthen buyer 3’s competitive intensity index by internalizing her free-riding possibilities on the high competitive intensities of agents in other value networks. Such stronger version of buyer 3’s competitive intensity, denoted by \(s_3^*\), equals \(s_3^* = \max_{i=1,2} \{u - (u - w_i^*)\} = u\) in this example, thus resulting in a sharp bound for buyer 3’s minimal payoff in the core, since \(u \leq \pi_3^{\text{min}} \leq \pi_3^{\text{max}} \leq u\). Note also that this sharpens the bound on firm \(b\)’s payoff in the core, since the so-called strong residual of \(b\) is \(s_{b}^* = v(\{b, 3\}) - s_3^* = 0\), effectively resolving the problem at hand: \(0 \leq \pi_b^{\text{min}} \leq \pi_b^{\text{max}} \leq 0\).

More generally, the weak competitive periphery of agent \(i\) under \(P^*\) considered only deviations by agent \(i\) to join a network that exists. Here we consider the possibility of joining one alternative value network while simultaneously excluding some players in that particular value network.
Suppose $i$ joins a value network $X \neq X_i^*$ as a replacement for some subset $Y \subset X$ that brings an added value of $av_Y(X)$ to $X$. Note that individually each agent $j \in Y$ cannot receive less than $w_j^*$ and jointly the group $Y$ cannot receive less than what they could produce on their own, namely $v(Y)$.\textsuperscript{6} As a result, the agents in $X \setminus Y$ have no more than

$$av_Y(X) - \max \left\{ \sum_{j \in Y} w_j^*, v(Y) \right\}$$

(10)

to somehow share amongst them, so expression (10) gives an upper bound on the opportunity cost that the agents in $X \setminus Y$ bear for not partnering with those in $Y$. Consider now the value added by agent $i$ to value network $X \setminus Y$ net of such an upper bound on the opportunity cost for the agents in $X \setminus Y$:

$$av_i(i \cup X \setminus Y) - \left( av_Y(X) - \max \left\{ \sum_{j \in Y} w_j^*, v(Y) \right\} \right).$$

(11)

Expression (11) gives a lower bound on how much the agents in $X$ can profit from replacing the agents in $Y \subset X$ with $i$.

The following notion of strong competitive intensity for agent $i$ is simply based on having the freedom to pick any $X \in \mathcal{P}^* \setminus X_i^*$ and any $Y \subset X$ to maximize the expression in (11).

**Definition 3.** The strong competitive intensity for $i$ in a focal value partition $\mathcal{P}^*$ is

$$s_i^* \equiv \max_{Y \subset X, \ X \in \mathcal{P}^* \setminus X_i^*} \left[ av_i(i \cup X \setminus Y) - \left( av_Y(X) - \max \left\{ \sum_{j \in Y} w_j^*, v(Y) \right\} \right) \right].$$

In a similar way we have:

**Definition 4.** The strong competitive intensity for $i$’s partners is

$$\sum_{j \in X_i^* \setminus i} s_j^*,$$

and the strong residual of $i$ under $\mathcal{P}^*$ is what is left from $v(X_i^*)$ after paying his partners their

\textsuperscript{6}Recall expression (5).
strong competitive intensity but no more than i’s added value to X_i, i.e.,
\[ s^*_i \equiv \min \left\{ v(X_i^*) - \sum_{j \in X_i^* \setminus i} s^*_j, a v_i(X_i^*) \right\}. \] (12)

Again, while agent i’s external alternatives create a lower bound on what i may get, the intensity of competition for i’s partners places an upper bound on i’s appropriation of the share of the value created within i’s own value network, which is complemented by the internal threat of those partners to exclude i from X_i. This leads to bounds that cannot be worse than those in Proposition 1.

Proposition 2. Given a focal value partition \( \mathcal{P}^* \), for all \( i \in N \),
\[ \pi^\text{min}_i \geq s^*_i \geq w^*_i, \quad \text{and} \]
\[ \pi^\text{max}_i \leq s^*_i - \pi^\text{min}_i. \] (13) (14)

Proof. Note from the definition of \( s^*_i \) that choosing \( Y = \emptyset \) would make \( s^*_i = w^*_i \), so allowing \( Y \) to differ from \( \emptyset \) implies that \( s^*_i \geq w^*_i \) and \( w^*_i \geq s^*_i \), so we just need to show that \( s^*_i \leq \pi^\text{min}_i \) and \( \pi^\text{max}_i \leq s^*_i \). To prove that \( \pi^\text{min}_i \geq s^*_i \), suppose to the contrary that \( s^*_i - \pi^\text{min}_i > 0 \) and assume that \( s^*_i > w^*_i \) to avoid triviality. Letting \( X^* \in \mathcal{P}^* \setminus X_i^* \) and \( Y^* \subset X^* \) be some maximizers of
\[ a v_i(i \cup X^* \setminus Y^*) = \left( a v_Y(X) - \max \left\{ \sum_{j \in Y} w^*_j, v(Y) \right\} \right), \]
we must have
\[ a v_i(i \cup X^* \setminus Y^*) - \left( a v_Y^*(X^*) - \max \left\{ \sum_{j \in Y^*} w^*_j, v(Y^*) \right\} \right) \leq \pi^\text{max}_i > 0. \]

This means that the agents in \( X^* \) could exclude a subset \( Y^* \subset X^* \) of agents and replace them with i, thus making themselves better off together with agent i. Because such a mutually profitable deviation cannot happen if the value distribution is consistent with competition,\(^7\) a contradiction is obtained and hence we have that \( \pi^\text{min}_i \geq s^*_i \).

\(^7\)In terms of costs and benefits of such a deviation, note that the agents in \( Y^* \) obtain no less than \( \max \left\{ \sum_{j \in Y^*} w^*_j, v(Y^*) \right\} \) out of the value \( a v_Y^*(X^*) \) they bring in to \( X^* \), so excluding them from \( X^* \) costs no
We conclude the proof by demonstrating that $\pi_i^{\text{max}} \leq s^*_i$. Letting $s^*_{-i} < w^*_i$ to avoid triviality, the proof parallels the one showing that $\pi_i^{\text{max}} \leq w^*_i$ in Proposition 1.

One could push the logic and provide sharper bounds for $\pi_i^{\text{min}}$ and $\pi_i^{\text{max}}$ by using $\{s^*_j\}_{j \in Y}$ instead of $\{w^*_j\}_{j \in Y}$ in (10) and repeat the outlined procedure. The output can then be used to further iterate this process. With each step $t = 0, 1, 2, \ldots$, we obtain a new element in a sequence $s^*_i(t)$, with $s^*_i(0) = w^*_i$. Since this sequence is monotonically increasing and bounded above by $\pi_i^{\text{min}}$, it has a limit that can be used to define the competitive intensity for $i$ and for $i$’s partners.

Conditional on using our approach towards characterizing bounds for competitive intervals, this algorithm provides the tightest possible bounds, although it need not always pin down the competitive interval of all agents. However in some situations, such as the differentiated product markets setting we consider below, no more than one round of iteration may be necessary to pin down the competitive interval of all agents.

### 3.2.3 Generalized weak competitive intensity

Above we considered only the option of a player $i$ to join an external group $X$ which was itself a value partition. Naturally agent $i$ can ensure himself an even better payoff if he has the freedom to choose from $N$ the group to separate with from $X^*_i$. We can then use the fact that no group can expect to receive more than the value they jointly add to the economy $av_S(N)$ and that in particular no individual player can get more than his marginal contribution $av_i(N)$. Define

$$C_i \equiv \arg \max_{S \subseteq N \setminus i} \left[ v(S \cup i) - \min \left\{ av_S(N), \sum_S av_j(N) \right\} \right].$$

It is perhaps not obvious why the expression above is the generalization of (6), but recall that in a value partition $\mathcal{P}$ we must have that $v(N) = \sum_{G \in \mathcal{P}} v(G)$ and therefore

$$v(G) = av_G(N) \text{ for any } G \in \mathcal{P}. \quad (15)$$

more than

$$av_Y^*(X^*) - \max \left\{ \sum_{j \in Y^*} w_j^*, v(Y^*) \right\},$$

whereas the opportunity cost of having $i$ join $X^* \setminus Y^*$ is $\pi_i^{\text{min}}$. So the benefit of replacing $Y^*$ with $i$ for the subset of agents $X^* \setminus Y^*$, which equals $av_i(\{i\} \cup X^* \setminus Y^*)$, always exceeds the opportunity costs of such an action, which cannot happen if $\pi_i^{\text{min}}$ is a payoff consistent with competition.
Also recall that for all $S \in \mathcal{P}^*$ we have that

$$\text{av}_{S}(N) \leq \sum_{S} \text{av}_{i}(N)$$

So the weak competitive periphery is a similar problem to the generalized competitive periphery when the maximizer is restricted to an $S \in \mathcal{P}^*$. This leads to the following definition

*The generalized weak competitive intensity for $i \in N$ is*

$$w_i \equiv v(C_i \cup i) - \min \left\{ \text{av}_{C_i}(N), \sum_{C_i} \text{av}_{j}(N) \right\}.$$  \hspace{1cm} (16)

Not only cannot any group of agents (which includes $i$) be jointly paid more than their joint marginal contribution to $v(N)$, but also each agent in any subset must be paid at least his generalized competitive intensity.

From (15), a natural extension of $\rho_{-i}^*$ to a situation with freedom to choose trading partners is the following.

*The weak generalized residual of $i$ is:*

$$\overline{w}_i \equiv \min_{S \subseteq N} \text{av}_{S \cup i}(N) - \sum_{S \setminus i} w_j.$$  

Denote the set of payoffs that are compatible with competitive intensity in a game $(N, v)$ by

$$CI = \left\{ \pi \in R^n | \sum_{N} \pi_i = v(N), \underline{w} \leq \pi \leq \overline{w} \right\}$$

where $\underline{w}$ and $\overline{w}$ are vectors with respectively elements $\underline{w}_i$ and $\overline{w}_i$. We then have:

**Proposition 3.** Given any focal value partition $\mathcal{P}^*$, for all $i \in N$, $w_{-i}^* \geq \overline{w}_i \geq \pi_{i}^{\text{max}} \geq \pi_{i}^{\text{min}} \geq \underline{w}_i \geq w_{i}^*$. Moreover the core is a subset of those payoffs that are compatible with generalized competitive intensity, i.e., $C \subseteq CI$.
Proof: If $\pi \in C$ then $\rho_i \leq \pi_i \leq \bar{\rho}_i$. Thus $\pi \in \{ \pi \in \mathbb{R}^n | \rho \leq \pi \leq \bar{\rho} \}$. Since both the elements of $CI$ and $C$ satisfy $\sum_N \pi_i = v(N)$, we have that $C \subseteq CI$.

So even a more general definition of competitive intensity, which allows us to bound how much a player gets, does not depend in a essential way on the value added by $i$ to his own value network $av_i(X_i^*)$. Again, this raises serious question on the conventional wisdom message that agents aiming to improve their payoffs should focus on increasing their value added to their current trading partners.

Note that if the core is nonempty then the set of payoffs that are compatible with competitive intensity is also non-empty. Moreover the extreme points of $CI$ can be described by pseudo marginal vectors. In addition these vectors can be used to infer if competitive intensity characterizes exactly the competitive bounds of a player.

Take an order of the agents $\sigma$, and let $\sigma(k)$ denote the agent in position $k$ in the order $\sigma$. The set of all possible orders of $N$ is given by $N^\sigma$. A pseudo marginal vector associated with an ordering $\sigma \in N^\sigma$ is an efficient vector that starts by paying the first player up his competitive intensity residual provided the remaining players receive at least their competitive intensity, and proceeds similarly with each subsequent agent from the residual surplus. The set of agents in $\sigma$ that receive their competitive intensity residual is called the head of $\sigma$, while the set of agents that receive only their competitive intensity are called the tail of $\sigma$. Formally:

The payoff vector $\pi^\sigma$ is the pseudo marginal vector associated with an ordering $\sigma \in N^\sigma$ if for each $i \in N$ we have

$$\pi^\sigma_{\sigma(k)} = \begin{cases} 
\rho_{\sigma(k)} & \text{if } f_{\sigma(k)} \leq v(N) \\
\bar{\rho}_i & \text{if } f_{\sigma(k)} \geq v(N) \\
v(N) - f_{\sigma(k)} & \text{otherwise},
\end{cases}$$

where

$$f_{\sigma(k)} = \sum_{j=1}^k w_{\sigma(j)} + \sum_{j=k+1}^n w_i.$$

Note that $CI$ is the convex hull of the pseudo marginal vectors, i.e.,

$$CI = \text{conv} \{ \pi^\sigma | \sigma \in N^\sigma \}.$$
In general there are less $\pi^\sigma$ than orderings, since any two vectors with the same head and tail are equivalent—indeed independent of the exact order of the players in the head or the tail.

These vectors can be used to access if some bound given by competitive intensity is tight. Notice that while it can be a quite challenging problem to determine the core or even the extreme payoffs, it is on the other hand a simple task to check if a particular vector belongs to the core. We have:

**Proposition 4.** If some $\pi^\sigma$ belongs to the core, then $w_i = \pi_i^{\text{max}}$ if $\pi_i^\sigma = w_i$ and $w_i = \pi_i^{\text{min}}$ if $\pi_i^\sigma = w_i$.

We now provide sufficient conditions for the competitive intervals to be captured by the concept of competitive intensity for every agent $i \in N$.

**Proposition 5.** The set of payoffs that are compatible with a generalized competitive intensity coincides with those that are consistent with competition, i.e., $CI = C$, if and only if for each $G \subseteq N$

$$v(G) \leq \max \left\{ \sum_{i \in G} w_i, v(N) - \sum_{i \in N \setminus G} w_i \right\}.$$  

(17)

If $CI = C$ then $w_i = \pi_i^{\text{max}}$ and $w_i = \pi_i^{\text{min}}$ for every $i \in N$.

The last Proposition carries the following important message. If competitive intensity is sufficiently strong, then both elements on the right hand side of (17) are high. This means that the inequalities will be satisfied for each $G \subseteq N$. Therefore competitive intensity alone fully characterizes the set of outcomes that are consistent with competition, and the respective competitive intervals, when competitive intensity is sufficiently strong.

### 4 A geometric model of horizontal and vertical differentiation

#### 4.1 Setup

We now introduce a model of competition in a two-sided market with both vertical and horizontal differentiation. In this model, products are located in the positive quadrant of the two-dimensional Euclidean plane. Consumer preferences are defined in such a way that the greater a product’s distance from the origin, the higher its quality (i.e., all consumers value greater distance more).\(^8\)

\(^8\)The following results also obtain in a traditional Hotelling duopoly setting, augmented to permit vertical differentiation. Our results are driven by the fact that buyers are ordered based upon their preference towards a focal firm, regardless of which firm is chosen as focal.
At the same time, two products that are equidistant from the origin are valued differently by different consumers. This setting allows us to apply the results of the preceding section to several interesting cases that arise within a familiar context.

The set of agents in the market is $N = N_F \cup N_B$, where $N_F \equiv \{f_1, \ldots, f_{n_F}\}$ is a set of single-product firms and $N_B \equiv \{b_1, \ldots, b_{n_B}\}$ is a set of unit-demand buyers. Assume $N_F \cap N_B = \emptyset$, $n_F \geq 1$, $n_B \geq 1$ and let $n \equiv n_F + n_B$. On the supply side, firms have the same constant marginal cost of production, normalized to zero. The industry capacity profile is $q = (q_1, \ldots, q_{n_F}) \in \mathbb{N}^{n_F}$, where $q_r$ denotes $f_r$’s installed capacity. Firm $f_r$’s product is represented by a bidimensional vector $\vec{x}_r \in \mathbb{R}^2$, the components of which represent measurable features of a product. Using polar coordinates, $\vec{x}_r$ is represented by $(l_r, \theta_r)$, where $l_r \in \mathbb{R}_+$ is the length of the vector and $\theta_r \in [0, 2\pi]$ is its angle in relation to the positive abscissas (“x”) axis.

On the demand side, buyer preferences are completely described by $\theta_i \in [0, 2\pi]$, that is, an angle in relation to the positive abscissas axis. Specifically, buyer $b_i$’s utility of consuming one unit of firm $r$’s product is given by:

$$U_i(\vec{x}_r) = \begin{cases} 0 & \text{if } \pi/2 \leq \angle_{ir} \leq 3\pi/2, \\ l_r \cos(t\angle_{ir}) & \text{otherwise} \end{cases},$$

where “$\angle_{ir}$” is the size of the smallest angle between $\theta_i$ and $\theta_r$ (measured in radians) and parameter $t \in [0, 1]$ captures the intensity of horizontal differences. Thus, given two buyers $i$ and $j$ with $\angle_{ir} > \angle_{jr}$, buyer $j$ places a higher value on firm $r$’s product than $j$ (at any quality level). Still, both buyers value larger values of $l_r$. Thus, $\theta_r$ is the horizontal differentiation parameter of firm $r$’s product (variety) and $l_r$ is its vertical differentiation parameter (quality).

The setup is graphically depicted in Figure 1. In this case, firm $r$’s product is located at $\vec{x}_r = (l_r, \theta_r)$. Buyer 1’s most preferred quality direction is along the ray at angle $\theta_1$. Buyer 1’s willingness-to-pay for $\vec{x}_r$ is the length of $\overline{AC}$. Buyer 2’s willingness-to-pay is the length of $\overline{AB}$. These lengths are determined by the perpendicular distance from $\vec{x}_r$ to the ray associated with the buyer’s most preferred direction. As $l_r$ increases, the utility of both buyers increases.

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9One of the novelties of our model is that horizontal and vertical effects interact in it. To the best of our knowledge, such interaction is not allowed in the most widely used models in the industrial organization literature (c.f. Hotelling 1929, Shaked and Sutton 1982 and Neven and Thiss 1990).

10Compute the scalar projection of $\vec{x}_r$ onto the ray of buyer 1’s most preferred direction so as to obtain the length of $\overline{AC}$ (i.e., $l_r \cos(\theta_r - \theta_1)$).
Figure 1: Demand mechanics – one product, two buyers, $t = 1$

By restricting attention to the positive quadrant, products always have nonnegative value to buyers. For those products whose vertical attributes are positively valued by a buyer, enhancing them definitely increases that buyer’s utility. However, vertical attributes become less important for the buyer as the product is farther away from her ideal direction. Consequently, any enhancement of vertical features is valued less as a buyer becomes more distant in her horizontal taste dimension. In the extreme case, a buyer $i$ located on the $x_2$ axis $(\theta_i = \pi/2)$ receives zero utility from a product $r$ located on the $x_1$ axis $(\theta_r = 0)$, regardless of its quality $l_r$.

4.2 Duopoly application

We consider the simplest competitive setting, so we let $n_F = 2$ and index firms by $a$ and $b$, with respective features $\vec{x}_a = (l_a, \theta_a)$ and $\vec{x}_b = (l_b, \theta_b)$. Throughout, we will be loose and refer to the length of the arc between an agent’s location in the circumference and the positive abscissas
axis as this agent’s distance (measured in radians) from such an axis. If we assume that firms are symmetric from a horizontal differentiation point of view and force them to interact in the product market, we can suppose without loss of generality that \( \theta_a = \pi/8 \) and \( \theta_b = 3\pi/8 \).

To maintain simplicity, we assume that there are \( n_B = 3 \) buyers who are equidistant from each other. Buyers are labeled 1, 2 and 3. In order for the setting to exhibit symmetric horizontal differentiation, there is no loss of generality in assuming that buyer 1’s distance (from the positive abscissas axis) is \( \theta_1 = \theta_a = \pi/8 \), buyer 2’s distance is \( \theta_2 = 2\pi/8 \), whereas buyer 3’s distance is \( \theta_3 = \theta_b = 3\pi/8 \).\(^{11}\)

The utility generated from purchasing one unit of the good sold by firm \( a \) for buyers 1, 2 and 3 are respectively equal to \( U_1(\vec{x}_a) = l_a, \ U_2(\vec{x}_a) = l_a \cos(t\pi/8) \) and \( U_3(\vec{x}_a) = l_a \cos(t\pi/4) \). Similarly, the utility generated from purchasing one unit of the good sold by firm \( b \) for buyers 1, 2 and 3 are respectively equal to \( U_1(\vec{x}_b) = l_b \cos(t\pi/4) \), \( U_2(\vec{x}_b) = l_b \cos(t\pi/8) \) and \( U_3(\vec{x}_b) = l_b \).

We assume that \( \cos(t\pi/4) < l_b/l_a \leq 1 \), so that firm \( a \) is (possibly) vertically superior to firm \( b \) and buyer 3 prefers firm \( b \) over \( a \) if confronted with the choice between both. This situation is represented in Figure . We recall that firm \( i \)’s capacity is denoted by \( q_i, i \in \{a, b\} \). Because we shall ultimately be interested in the impact of vertical differences among firms, one can think of \( l_b/l_a < 1 \) as being a working assumption, with \( \lim_{l_b/l_a \to 1} l_b/l_a \) corresponding to the case in which there are no vertical differences among firms. If \( l_b/l_a < 1 \), it is worth noting for \( q_a + q_b \geq 3 \) that there always exists a unique value partition that is not trivial.

It is relatively straightforward to verify that an application of Proposition 1 yields sharp bounds for the firms’ competitive intervals when \( (q_a, q_b) \in \{(1, 2), (2, 1), (3, 2), (3, 3)\} \).

**Lemma 1.** Suppose that:

(i) \( q_a = 1 \) and \( q_b = 2 \): Then \( \pi_a^{\text{min}} = (l_a - l_b) \cos(t\pi/8), \ \pi_a^{\text{max}} = l_a, \ \pi_b^{\text{min}} = 0, \ \text{and} \ \pi_b^{\text{max}} = l_b(1 + \cos(t\pi/8)). \)

(ii) \( q_a = 2 \) and \( q_b = 1 \): Then \( \pi_a^{\text{min}} = 0, \ \pi_a^{\text{max}} = l_a(1 + \cos(t\pi/8)), \ \pi_b^{\text{min}} = 0, \ \text{and} \ \pi_b^{\text{max}} = l_b. \)

(iii) \( q_a = 3 \) and \( q_b = 2 \): Then \( \pi_a^{\text{min}} = 0, \ \pi_a^{\text{max}} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)), \ \pi_b^{\text{min}} = 0, \ \text{and} \ \pi_b^{\text{max}} = l_b - l_a \cos(t\pi/4). \)

(iv) \( q_a = 3 \) and \( q_b = 3 \): Then \( \pi_a^{\text{min}} = 0, \ \pi_a^{\text{max}} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)), \ \pi_b^{\text{min}} = 0, \ \text{and} \ \pi_b^{\text{max}} = l_b - l_a \cos(t\pi/4). \)

\(^{11}\)Buyer 2 is then at a symmetric distance from each firm, whereas buyer 1 is at the same distance from firm \( a \) as is buyer 3 from firm \( b \) (i.e., zero).
Figure 2: Duopoly with three buyers, $t = 1$

When $(q_a, q_b) \in \{(2, 2), (2, 3)\}$, Proposition 1 pins down the exact values of the core payoff for firm $a$, but it fails to do it for firm $b$. The assumption that firm $a$ is vertically superior makes it produce at capacity, but its full capacity utilization turns it into a weak competitor vis-à-vis firm $b$, which could transact with any of firm $a$’s buyers if any of them unilaterally wanted to. Suppose that $w_3^* = 0$ were equal to $\pi_3^{\text{min}}$. In such a case, despite buyer 3 transacts with firm $b$, the resulting weakness of firm $a$ would make it easy for such a buyer to offer herself as a replacement for buyer 2. Because both firm $a$ and buyer 3 would benefit from such an arrangement, we cannot have $\pi_3^{\text{min}} = 0$, but rather $\pi_3^{\text{min}} > 0$. Because buyer 3’s competitive intensity must be positive, we must have that $\pi_b^{\text{max}} < l_b$, where $l_b$ is equal to the weak residual of firm $b$. The slackness of this bound can be fixed by applying now Proposition 2, which yields sharp bounds for firm $b$’s competitive intensity.

\[\text{That the weak competitive intensity index fails in providing an exact bound when firm } b \text{ has idle capacity but firm } a \text{ does not is certainly not a coincidence.}\]
intervals when \((q_a, q_b) \in \{(2, 2), (2, 3)\} \).

**Lemma 2.** Suppose that:

(i) \(q_a = 2\) and \(q_b = 2\): Then \(\pi_a^{\min} = 0\), \(\pi_a^{\max} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\), \(\pi_b^{\min} = 0\), and \(\pi_b^{\max} = l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8)\).

(ii) \(q_a = 2\) and \(q_b = 3\): Then \(\pi_a^{\min} = 0\), \(\pi_a^{\max} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\), \(\pi_b^{\min} = 0\), and \(\pi_b^{\max} = l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8)\).

Tables 1 and 2 summarize the previous Lemmata.

<table>
<thead>
<tr>
<th>((q_a, q_b))</th>
<th>Firm a’s competitive interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 1)</td>
<td>([0, l_a(1 + \cos(t\pi/8))])</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>([(l_a - l_b) \cos(t\pi/8), l_a])</td>
</tr>
<tr>
<td>(2, 2), (2, 3), (3, 2), (3, 3)</td>
<td>([0, l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))])</td>
</tr>
</tbody>
</table>

Table 1: Firm a’s competitive interval as a function of \(q_a\) and \(q_b\)

<table>
<thead>
<tr>
<th>((q_a, q_b))</th>
<th>Firm b’s competitive interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 1)</td>
<td>([0, l_b])</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>([0, l_b(1 + \cos(t\pi/8))])</td>
</tr>
<tr>
<td>(2, 2), (2, 3)</td>
<td>([0, l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8)])</td>
</tr>
<tr>
<td>(3, 2), (3, 3)</td>
<td>([0, l_b - l_a \cos(t\pi/4)])</td>
</tr>
</tbody>
</table>

Table 2: Firm b’s competitive interval as a function of \(q_a\) and \(q_b\)

We have just shown that there is therefore no need to solve the complicated linear optimization problem with \(2^5 - 1 = 31\) constraints for each capacity vector, since the bounds we derived using Propositions 1 and 2 are sharp. This allows us to easily draw a number of insights on which we elaborate next.

### 4.3 Managerial implications

**Redefining Competitive Advantage**

Notice from Tables 1 and 2 that the vertical superiority of firm \(a\) over \(b\) need not result in greater profitability, as happens for instance if \((q_a, q_b) = (2, 2)\).
In its worst-case scenario, firm \(a\) can be seen to compete for rents with buyer \(i \in \{1, 2\}\) in a bilateral monopoly situation. If firm \(a\)’s bargaining skills are very poor, say, then its lack of a credible threat makes a weak target for its buyers, who may well appropriate all the surplus generated by their transaction with the firm. In the best-case scenario for firm \(a\) (say its bargaining skills are excellent), it can extract all the surplus from buyer \(i \in \{1, 2\}\) after discounting the credible threat that she has.\(^{13}\) So in a situation in which firm \(a\) has no bargaining skills whatsoever and firm \(b\) barely has some, we may well have that \(\pi_a = 0 < \pi_b = \varepsilon\) for some small \(\varepsilon > 0\), even though the difference between \(l_a\) and \(l_b\) is not negligible.\(^{14}\) This casts a doubt on traditional definitions of competitive advantage (which many would have defined a priori as \(l_a - l_b\) in the case at hand, given that firms are symmetric otherwise). Of course, when both firms are in their best-case scenarios and firms’ payoffs are determined by the competitive intensity for the respective partners, firm \(a\) does outperform firm \(b\).

**Strategic Commitments and Their Effects**

The previous discussion brings us to the importance of strategic commitments arising from sunk investments (Ghemawat 1991) when it comes to defining competitive advantage. In the current case, even if marginal cost does not increase with greater production, larger capacity utilization may result in a strategic disadvantage because firm \(a\) weakens its bargaining position vis-à-vis buyers. Thus, in our setting firm \(a\) cannot credibly threaten to stop serving buyer 2 and serve buyer 3, since firm \(b\) is stronger when it comes to competing for buyer 3. Also, the most powerful threat available to firm \(a\) against buyer 1 is to serve buyer 2. Yet, the credibility of this threat is undermined by the large capacity of firm \(a\), which induces to already serve buyer 2. Buyers, however, do have credible threats because firm \(b\) has idle capacity. So the example we have just examined illustrates the indirect effects that may ensue from committing to certain courses of action, which takes us to the next issue.

**Relationship between Added Value and Firm Performance**

Enhancing added value may actually destroy profitability. Starting from \((q_a, q_b) = (1, 2)\), Table 2 shows that an increase in firm \(a\)’s capacity by one unit (weakly) shifts firm \(b\)’s payoff interval to the left.\(^{15}\) Table 1 shows

\(^{13}\)Firm \(a\) extracts \(l_a - l_b \cos(t \pi / 4)\) from buyer 1 and \((l_a - l_b) \cos(t \pi / 8)\) from buyer 2.

\(^{14}\)When \((q_a, q_b) = (2, 2)\), there certainly exists a distribution of value consistent with competition in which \(\pi_a = 0\) and \(\pi_b = \varepsilon\), as shown in the Appendix.

\(^{15}\)This happens because it always holds that \(l_a \cos(t \pi / 4) > (l_a - l_b) \cos(t \pi / 8)\) (see (20)), which can be equivalently written as \(\frac{l_a}{l_b} < 1 + \frac{\cos(t \pi / 4)}{\cos(t \pi / 8) - \cos(t \pi / 4)}\).
that such a capacity expansion (weakly) shifts firm a’s payoff interval to the left if

$$\frac{l_a}{l_b} \leq 1 + \frac{\cos(t\pi/4)}{\cos(t\pi/8)}$$

(19)

(which for instance holds when firms are close to being symmetric or if horizontal differences are not very relevant, so that competition is intense in any of the cases). Therefore, increasing buyers’ added values may indeed lead to lower profits. Adding capacity has a positive "value-creation effect," since it allows firm a to steal buyer 2 away from b given that $l_a > l_b$. If competitive intensities remain fixed, this effect should be weighed against the cost of adding capacity, and the decision would be straightforward. However, there is another effect at play because competitive intensities do not remain fixed but actually change. We refer to this effect that arises from the change in competitive intensities as the "value-appropriation effect." In the current case, the value-appropriation effect of capacity expansion by firm a is negative for such a firm because adding capacity leaves a in a worse bargaining position vis-à-vis buyers when it comes to appropriating value. Such a weaker position happens because adding capacity relaxes firm b’s capacity constraint and also undermines the credibility of firm a’s threat to replace buyers with some other ones. Therefore, normative admonitions to firms that they increase their added values should be made with care: the firm may not be able to appropriate such an increase, and in fact it may be left in a weaker position from a strategic point of view. Our framework provides a powerful lens through which to understand the sign of value-appropriation effects and thus view whether they conflict or reinforce value-creation effects, thus allowing for more informed strategic decision-making.

**Capacity Expansion Strategies** Let us now add a stage prior to product market interaction in which firms a and b noncooperatively choose $q_a$ and $q_b$ in a simultaneous fashion. Instead of characterizing the equilibrium outcome of the resulting biform game, we will focus on ruling out some outcomes that cannot arise in equilibrium under extremely weak conditions (i.e., regardless of the values taken by appropriation factors). In particular, let us assume that having one more unit of capacity requires incurring sunk cost $\kappa$, where $\kappa > 0$ may be arbitrarily small. More importantly, we will also assume that a firm has some positive willingness to pay for an extra unit of capacity only if there is some possibility that its subsequent payoff will be greater. If there is no possibility that such a payoff will be greater, the positive cost of adding one more unit of capacity will induce

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16Recalling that $t \in [0, 1]$, note that $1 + \frac{\cos(2x)}{\cos(x)}$ is decreasing in $x$ for $x \in [0, \pi/8]$. 

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the firm not to add it.

In the light of the results highlighted in Tables 1 and 2, neither \((q_a, q_b) = (3, 2)\) nor \((q_a, q_b) = (3, 3)\) can arise as equilibrium outcomes in this scenario. The reason is that the interval for firm \(a\)'s payoff is the same regardless of whether \((q_a, q_b) = (2, 2), (q_a, q_b) = (3, 2), (q_a, q_b) = (2, 3)\) or \((q_a, q_b) = (3, 3)\). In addition, we have that \((q_a, q_b) = (2, 3)\) cannot arise in equilibrium either, since the interval for firm \(b\)'s payoff is the same regardless of whether \((q_a, q_b) = (2, 2)\) or \((q_a, q_b) = (2, 3)\). We cannot have \((q_a, q_b) = (2, 2)\) either because firm \(b\) could leave its minimal payoff unchanged and its maximal payoff larger by unilaterally installing one unit of capacity less. One can similarly rule out \((q_a, q_b) = (3, 1)\) and \((q_a, q_b) = (1, 3)\). Also, we can also discard \((q_a, q_b) = (1, 2)\) sometimes (but not always): in particular, if it holds that \(\frac{l_a}{l_b} \leq 1 + \frac{\cos(t\pi/4)}{\cos(t\pi/8)}\) (see expression (19)). We are then left with the following candidates for equilibrium outcomes: \((q_a, q_b) = (3, 0), (q_a, q_b) = (0, 3), (q_a, q_b) = (2, 1), (q_a, q_b) = (1, 2)\) for \(\frac{l_a}{l_b} > 1 + \frac{\cos(t\pi/4)}{\cos(t\pi/8)}\) \(^{17}\) \((q_a, q_b) = (1, 1), (q_a, q_b) = (1, 0), (q_a, q_b) = (0, 1)\) and \((q_a, q_b) = (0, 0)\). Out of this set of equilibrium outcome candidates, which one is played depends on the value of \(\kappa\) and on behavioral assumptions about how firms view appropriation of subsequent payoffs. However, it is worthwhile mentioning that, regardless of which such assumptions are made, there will never be excess capacity in the market (as Kreps and Scheinkman 1983 showed for homogeneous goods with divisible capacity), a result that is likely to extend well beyond the example we consider.

5 Conclusion

\[\text{[Forthcoming]}\]

\(^{17}\)Recall that addition of one extra unit of capacity by firm \(a\) would decrease its maximal payoff if expression (19) holds.
REFERENCES


Williamson, O. E. (1971). The Vertical Integration of Production: Market Failure Considera-
APPENDIX

Proof of Lemmas 1 and 2. **Cases in which** \((q_a, q_b) = (1, 2)\)  
Let us look first at the cases in which \((q_a, q_b) = (1, 2)\), so that \(v(N) = l_a + l_b + l_b \cos(t\pi/8)\). Let \(u_i \; (i \in \{1, 2, 3\})\) denote the payoff of buyer \(i\) in a core allocation. Similarly, let \(\pi_j \; (j \in \{a, b\})\) denote the payoff of firm \(j\) in a core allocation. Recalling that \(\pi_a + \pi_b + u_1 + u_2 + u_3 = v(N)\), expression (3) used for all \(G \subset N = \{1, 2, 3, a, b\}\) yields the following inequality constraints that \((u_1, u_2, u_3, \pi_a, \pi_b)\) must satisfy:

\[
\begin{align*}
0 & \leq \pi_a \leq l_a \\
0 & \leq \pi_b \leq l_b + l_b \cos(t\pi/8) \\
0 & \leq \pi_a + \pi_b \leq l_a + l_b + l_b \cos(t\pi/8) \\
\end{align*}
\]

\[
\begin{align*}
l_a & \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_b \cos(t\pi/8) \\
l_a \cos(t\pi/8) & \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_b \cos(t\pi/8) \\
l_b & \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \\
\end{align*}
\]

\[
\begin{align*}
l_a + l_b \cos(t\pi/8) & \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_b \cos(t\pi/8) \\
l_a + l_b & \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \\
l_a \cos(t\pi/8) + l_b & \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \\
\end{align*}
\]

\[
\begin{align*}
l_a & \leq \pi_a + u_1 \leq l_a \\
l_a \cos(t\pi/8) & \leq \pi_a + u_2 \leq l_a + l_b \cos(t\pi/8) - l_b \cos(t\pi/4) \\
l_a \cos(t\pi/4) & \leq \pi_a + u_3 \leq l_a + l_b - l_b \cos(t\pi/4) \\
\end{align*}
\]
\[ l_a \leq \pi_a + u_1 + u_2 \leq l_a + l_b \cos(t\pi/8) \]
\[ l_a \leq \pi_a + u_1 + u_3 \leq l_a + l_b \]
\[ l_a \cos(t\pi/8) \leq \pi_a + u_2 + u_3 \leq l_a + l_b \cos(t\pi/8) + l_b - l_b \cos(t\pi/4) \]
\[ l_a \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \]

\[ l_b \cos(t\pi/4) \leq \pi_b + u_1 \leq l_a + l_b + l_b \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_b \cos(t\pi/8) \]
\[ l_b \leq \pi_b + u_3 \leq l_b + l_b \cos(t\pi/8) \]

\[ l_b \cos(t\pi/8) + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_2 \leq l_a + l_b + l_b \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_b \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \]

\[ 0 \leq u_1 \leq l_a + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) \]
\[ 0 \leq u_2 \leq l_b \cos(t\pi/8) \]
\[ 0 \leq u_3 \leq l_b \]

\[ 0 \leq u_1 + u_2 \leq l_a + l_b \cos(t\pi/8) \]
\[ 0 \leq u_1 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) \]
\[ 0 \leq u_2 + u_3 \leq l_b + l_b \cos(t\pi/8) \]
\[ 0 \leq u_1 + u_2 + u_3 \leq l_a + l_b + l_b \cos(t\pi/8) \]

If we denote any nontrivial value partition given capacity vector \((q_a, q_b)\) by \(P_{q_a, q_b}^*\), we have that the unique nontrivial partition given \((q_a, q_b) = (1, 2)\) is \(P_{1,2}^* = \{\{a, 1\}, \{b, 2, 3\}\}\). Under \(P_{1,2}^*\), we now proceed to show that the weak competitive intensities/residuals, \(w_a^* = (l_a - l_b) \cos(t\pi/8)\),
\( w^*_a = l_a, w^*_b = 0, \) and \( w^*_{-b} = l_b(1 + \cos(t\pi/8)) \), are sharp bounds. To do so, we assume that we indeed have either \( \pi_a^{\text{min}} = (l_a - l_b)\cos(t\pi/8) \) or \( \pi_a^{\text{max}} = l_a \) or \( \pi_b^{\text{min}} = 0 \) or \( \pi_b^{\text{max}} = l_b(1 + \cos(t\pi/8)) \) and find no violation of the conditions imposed by the set of inequality constraints above. This will show that the bounds are giving us admissible core allocations, and hence they must be exact, since they were shown to bound below/above an agent’s competitive interval.

Thus, using the above inequalities that define the core of this game, it can be shown that \( \pi_a^{\text{min}} = (l_a - l_b)\cos(t\pi/8) \) can arise as a feasible core allocation for firm \( a \) whenever it holds that

\[
 l_a \cos(t\pi/4) - (l_a - l_b)\cos(t\pi/8) > 0, \tag{20}
\]

an inequality that is always satisfied given our assumption that \( l_b/l_a > \cos(t\pi/4) \). The core payoffs for the other players are \( u_1 = l_a - (l_a - l_b)\cos(t\pi/8), \ u_2 = l_b\cos(t\pi/8), \) and any pair of values \( (\pi_b, u_3) \) such that \( 0 \leq \pi_b \leq l_b - l_a\cos(t\pi/4) + (l_a - l_b)\cos(t\pi/8), \ l_a\cos(t\pi/4) - (l_a - l_b)\cos(t\pi/8) \leq u_3 \leq l_b \) and \( \pi_b + u_3 = l_b \) hold. Hence, our assumption that \( \cos(t\pi/4) < l_b/l_a \leq 1 \) implies that the lower bound on firm \( a \)'s payoff is sharp and coincides with the minimal payoff it can obtain.

Using the inequalities that define the core of this game, it can also be readily verified that \( \pi_a^{\text{max}} = l_a \) can arise as a feasible core allocation for firm \( a \). The core payoffs for the other players are \( u_1 = 0, \) and any tuple of values \( (\pi_b, u_2, u_3) \) such that \( l_b\cos(t\pi/4) \leq \pi_b \leq l_b(1 + \cos(t\pi/8)) \), \( 0 \leq u_2 \leq l_b(\cos(t\pi/8) - \cos(t\pi/4)) \), \( 0 \leq u_3 \leq l_b(1 - \cos(t\pi/4)) \), \( l_b(\cos(t\pi/8) + \cos(t\pi/4)) \leq \pi_b + u_2 \leq l_b(1 + \cos(t\pi/8)) \), \( l_b(1 + \cos(t\pi/4)) \leq \pi_b + u_3 \leq l_b(1 + \cos(t\pi/8)) \), and \( \pi_b + u_2 + u_3 = l_b(1 + \cos(t\pi/8)) \). Hence, our assumption that \( \cos(t\pi/4) < l_b/l_a \leq 1 \) implies that the upper bound on firm \( a \)'s payoff is sharp and coincides with the maximal payoff it can obtain.

Using the inequalities that define the core of this game, it can further be shown that \( \pi_b^{\text{min}} = 0 \) can arise as a feasible core allocation for firm \( b \). The core payoffs for the other players are \( u_2 = l_b\cos(t\pi/8), \ u_3 = l_b, \) and any pair of values \( (\pi_a, u_1) \) such that \( (l_a - l_b)\cos(t\pi/8) \leq \pi_a \leq l_a - l_b\cos(t\pi/4), \ l_b\cos(t\pi/4) \leq u_1 \leq l_a - (l_a - l_b)\cos(t\pi/8) \) and \( \pi_a + u_1 = l_a \) hold. Hence, our assumption that \( \cos(t\pi/4) < l_b/l_a \leq 1 \) implies that the lower bound on firm \( b \)'s payoff is sharp and coincides with the maximal payoff it can obtain.

\[\text{To show that } l_a \cos(t\pi/4) + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) > 0, \]

note that \( l_b/l_a > \cos(t\pi/4) \) yields that

\[
 l_a \cos(t\pi/4) + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) > l_a[\cos(t\pi/4) - (1 - \cos(t\pi/4))\cos(t\pi/8)],
\]

so the fact that \( \cos(2x) - (1 - \cos(2x))\cos(x) > 0 \) for \( x \in [0, \pi/8] \) (recall that \( t \in [0, 1] \)) shows that the initial claim is correct.
coincides with the minimal payoff it can obtain.

Using the inequalities that define the core of this game, it can finally be shown that $\pi_{b}^{\text{max}} = l_b(1 + \cos(t\pi/8))$ can arise as a feasible core allocation for firm $b$. The core payoffs for the other players are $u_2 = 0$, $u_3 = 0$, and any pair of values $(\pi_a, u_1)$ such that $l_a \cos(t\pi/8) \leq \pi_a \leq l_a$, $0 \leq u_1 \leq l_a(1 - \cos(t\pi/8))$, and $\pi_a + u_1 = l_a$. Hence, our assumption that $\cos(t\pi/4) < l_b/l_a \leq 1$ implies that the upper bound on firm $b$’s payoff is sharp and coincides with the maximal payoff it can obtain.

**Cases in which** $(q_a, q_b) = (2, 2)$ We consider next how the bounds work when firm $a$ adds one unit of capacity, so let now $q_a = 2$ and $q_b = 2$. Given that $v(N) = l_a + l_b + l_a \cos(t\pi/8)$, the conditions that $(u_1, u_2, u_3, \pi_a, \pi_b)$ must satisfy are:

\[
\begin{align*}
0 & \leq \pi_a \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \\
0 & \leq \pi_b \leq l_b \\
0 & \leq \pi_a + \pi_b \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_a & \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_a \cos(t\pi/8) & \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_b & \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_a + l_b \cos(t\pi/8) & \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_a + l_b & \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_b + l_a \cos(t\pi/8) & \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \\
l_a & \leq \pi_a + u_1 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \\
l_a \cos(t\pi/8) & \leq \pi_a + u_2 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \\
l_a \cos(t\pi/4) & \leq \pi_a + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) - l_b \cos(t\pi/4) \\
\end{align*}
\]
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \]
\[ l_a + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

\[ l_b \cos(t\pi/4) \leq \pi_b + u_1 \leq l_a + l_b - l_a \cos(t\pi/4) \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b \leq \pi_b + u_3 \leq l_b \]

\[ l_b \cos(t\pi/8) + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_3 \leq l_a + l_b \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

The unique nontrivial partition given \((q_a, q_b) = (2, 2)\) is \(P_{2,2}^* = \{\{a, 1, 2\}, \{b, 3\}\}\.\) Under \(P_{2,2}^*\), do we get sharp bounds using \(w_a^* = 0, w_b^* = 0, w_{-a}^* = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)),\) and \(w_{-b} = l_b\)? As we show next, the answer is affirmative, except for \(\pi_b^{\text{max}} = l_b\), in which case
the upper bound on firm b’s maximal payoff is not sharp because \( \pi_b = l_b \) cannot satisfy the set of inequality constraints above.

Thus, using the inequalities that define the core of this game, it can be shown that \( \pi_a^{\min} = 0 \) can arise as a feasible core allocation for firm a. The core payoffs for the other players are \( u_1 = l_a, \ u_2 = l_a \cos(t\pi/8), \) and any pair of values \( (\pi_b, u_3) \) such that \( 0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4), \ l_a \cos(t\pi/4) \leq u_3 \leq l_b \) and \( \pi_b + u_3 = l_b \) hold. Hence, our assumption that \( \cos(t\pi/4) < l_b/l_a \leq 1 \) implies that the lower bound on firm a’s payoff is sharp and coincides with the minimal payoff it can obtain.

Likewise, using the inequalities that define the core of this game, it can be shown that \( \pi_a^{\max} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)) \) and \( \pi_b^{\min} = 0 \) can arise as feasible core allocations for firms a and b. The core payoffs for the other players are \( u_1 = l_b \cos(t\pi/4), \ u_2 = l_b \cos(t\pi/8) \) and \( u_3 = l_b \). Hence, our assumption that \( \cos(t\pi/4) < l_b/l_a \leq 1 \) implies that the upper bound on firm a’s payoff and the lower bound on firm b’s payoff are sharp.

It can finally be shown that the inequalities that should hold in order for \( \pi_b^{\max} = l_b \) to arise as a core allocation for firm b would be violated, so we must have \( \pi_b^{\max} < l_b \).\(^{19}\) We then have that the upper bound in not tight because buyer 3’s weak competitive intensity fails to capture a credible threat that such a buyer has.

Such a credible threat is to offer herself to firm a as a replacement for buyer 2 if she is getting nothing from firm b. This will work as long as \( l_a \cos(t\pi/8) - u_2 < l_a \cos(t\pi/4) \), that is, as long as \( u_2 > l_a(\cos(t\pi/8) - \cos(t\pi/4)) \). So we must have \( u_2 \leq l_a(\cos(t\pi/8) - \cos(t\pi/4)) \) in order for firm a not to have an incentive to replace buyer 2 with 3. But if \( u_2 \leq l_a(\cos(t\pi/8) - \cos(t\pi/4)) \), the fact that \( l_a \cos(t\pi/4) > (l_a - l_b) \cos(t\pi/8) \) (recall expression (20)) implies that firm b could offer buyer 2 some amount between \( l_a(\cos(t\pi/8) - \cos(t\pi/4)) \) and \( l_b \cos(t\pi/8) \) and be better off. Buyer 3 exploits this tension to offer \( u_2 > l_a(\cos(t\pi/8) - \cos(t\pi/4)) \) to buyer 2 in order to show firm b that it could credibly replace buyer 2. In other words, buyer 3 can exploit to her benefit the fact that buyer 2’s competitive intensity equals \( l_b \cos(t\pi/8) \). This free riding on buyer 2’s positive competitive intensity whenever \( l_a \cos(t\pi/4) > (l_a - l_b) \cos(t\pi/8) \) is not internalized in the weak

\(^{19}\)This happens because we cannot possibly have \( l_b \leq \pi_b + u_3 \leq l_b \) (see (22)), \( l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \) (see (21)) and \( l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \) (see 23) be satisfied when \( \pi_b = l_b \). Note that \( l_b \leq \pi_b + u_3 \leq l_b \) and \( \pi_b = l_b \) would imply that \( u_3 = 0 \) and hence the other constraints would become \( l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \) and \( l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) \), which cannot possibly be fulfilled at the same time (recall that condition (20) always holds). What happens is that buyer 2 could always threaten to join the coalition formed by firm b and buyer 3, compensate firm b with \( l_b \) and ensure herself a payoff of \( l_b \cos(t\pi/8) \). However, if firm b and buyer 2 are respectively getting \( l_b \) and no less than \( l_b \cos(t\pi/8) \), then the grand coalition could exclude them and be better off: what the grand coalition loses by excluding them, \( l_a \cos(t\pi/8) + l_b - l_a \cos(t\pi/4) \), is less than their joint reward, which is no less than \( l_b + l_b \cos(t\pi/8) \).
competitive intensity index. This happens because firm $a$ is at full capacity but firm $b$ is not. In these cases, we will need to use the strong notion of competitive intensity for the firm with idle capacity.

In the current case, $s^*_a = l_a \cos(t \pi/4) - (l_a \cos(t \pi/8) - w^*_2)$, where $w^*_2 = l_b \cos(t \pi/8)$, so $s^*_b = l_a \cos(t \pi/4) - (l_a - l_b) \cos(t \pi/8)$, which is positive by expression (20). This would lead to an upper bound on firm $b$’s maximal payoff equal to $s^*_b = l_b - l_a \cos(t \pi/4) + (l_a - l_b) \cos(t \pi/8)$. It can indeed be shown now that $\pi^b_{\text{max}} = l_b - l_a \cos(t \pi/4) + (l_a - l_b) \cos(t \pi/8)$ can arise as a feasible core allocation for firm $b$. The core payoffs for the other players are $u_2 = l_b \cos(t \pi/8)$, $u_3 = l_a \cos(t \pi/4) - (l_a - l_b) \cos(t \pi/8)$, and any pair of values $(\pi_a, u_1)$ such that $(l_a - l_b) \cos(t \pi/8) + (l_a - l_b) \cos(t \pi/8) \leq \pi_a \leq l_a + (l_a - l_b) \cos(t \pi/8)$, $l_b \cos(t \pi/4) \leq u_1 \leq l_a - (l_a - l_b) \cos(t \pi/8)$ and $\pi_a + u_1 = l_a + (l_a - l_b) \cos(t \pi/8)$ hold. Hence, our assumption that $\cos(t \pi/4) < l_b/l_a \leq 1$ implies that the upper bound on firm $b$’s maximal payoff that arises from the strong notion of competitive intensity is sharp.

**Cases in which** $(q_a, q_b) = (2, 1)$ We now focus on the situations in which $(q_a, q_b) = (2, 1)$. In such a case, we have that $v(N) = l_a + l_b + l_a \cos(t \pi/8)$. The conditions that $(u_1, u_2, u_3, \pi_a, \pi_b)$ must satisfy are:

\[
\begin{align*}
0 & \leq \pi_a \leq l_a + l_a \cos(t \pi/8) \\
0 & \leq \pi_b \leq l_b \\
0 & \leq \pi_a + \pi_b \leq l_a + l_b + l_a \cos(t \pi/8) \\
\end{align*}
\]

\[
\begin{align*}
l_a & \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_a \cos(t \pi/8) \\
l_a \cos(t \pi/8) & \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_a \cos(t \pi/8) \\
l_b & \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_a \cos(t \pi/8) \\
\end{align*}
\]

\[
\begin{align*}
l_a + l_a \cos(t \pi/8) & \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t \pi/8) \\
l_a + l_b & \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_a \cos(t \pi/8) \\
l_a \cos(t \pi/8) + l_b & \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_a \cos(t \pi/8) \\
\end{align*}
\]
\[ \begin{align*}
  l_a & \leq \pi_a + u_1 \leq l_a + l_a \cos(t\pi/8) \\
  l_a \cos(t\pi/8) & \leq \pi_a + u_2 \leq l_a + l_a \cos(t\pi/8) \\
  l_a \cos(t\pi/4) & \leq \pi_a + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \\
  l_a + l_a \cos(t\pi/8) & \leq \pi_a + u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \\
  l_a + l_a \cos(t\pi/4) & \leq \pi_a + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \\
  l_a \cos(t\pi/8) + l_a \cos(t\pi/4) & \leq \pi_a + u_2 + u_3 \leq l_a + l_a \cos(t\pi/8) + l_b - l_b \cos(t\pi/4) \\
  l_a + l_a \cos(t\pi/8) & \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \\
  l_b \cos(t\pi/4) & \leq \pi_b + u_1 \leq l_a + l_b - l_a \cos(t\pi/4) \\
  l_b \cos(t\pi/8) & \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \\
  l_b & \leq \pi_b + u_3 \leq l_b \\
  l_b \cos(t\pi/8) & \leq \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \\
  l_b & \leq \pi_b + u_3 \leq l_a + l_b \\
  l_b \cos(t\pi/8) & \leq \pi_b + u_2 + u_3 \leq l_a + l_a \cos(t\pi/8) \\
  l_b & \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \\
  0 & \leq u_1 \leq l_a \\
  0 & \leq u_2 \leq l_a \cos(t\pi/8) \\
  0 & \leq u_3 \leq l_b \\
  0 & \leq u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \\
  0 & \leq u_1 + u_3 \leq l_a + l_b \\
  0 & \leq u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \\
  0 & \leq u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\end{align*} \]
The unique nontrivial partition given $(q_a, q_b) = (2, 1)$ is $P^*_{2,1} = \{\{a,1,2\}, \{b,3\}\}$. Under $P^*_{2,1}$, we now show that we get sharp bounds for $\pi^\text{min}_a$, $\pi^\text{min}_b$, $\pi^\text{max}_a$ and $\pi^\text{max}_b$ respectively using $w^*_a = 0$, $w^*_b = 0$, $w^*_{-a} = l_a(1 + \cos(t\pi/8))$, and $w^*_{-b} = l_b$.

Thus, using the inequalities that define the core of this game, it can be shown that $\pi^\text{min}_a = 0$ can arise as a feasible core allocation for firm $a$. The core payoffs for the other players are $u_1 = l_a$, $u_2 = l_a \cos(t\pi/8)$, and any pair of values $(\pi_b, u_3)$ such that $0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4)$, $l_a \cos(t\pi/4) \leq u_3 \leq l_b$ and $\pi_b + u_3 = l_b$ hold. Hence, our assumption that $\cos(t\pi/4) < l_b/l_a \leq 1$ implies that the lower bound on firm $a$’s payoff is sharp and coincides with the minimal payoff it can obtain. Indeed, this also proves that $\pi^\text{min}_b = 0$ is a sharp bound for firm $b$’s minimal payoff.

Using the inequalities that define the core of this game, it can also be shown that $\pi^\text{max}_a = l_a(1 + \cos(t\pi/8))$ can arise as a feasible core allocation for firm $a$. The core payoffs for the other players are $u_1 = 0$, $u_2 = 0$, and any tuple of values $(\pi_b, u_3)$ such that $0 \leq \pi_b \leq l_b - l_a \cos(t\pi/8)$, $0 \leq u_3 \leq l_b - l_a \cos(t\pi/8)$, and $\pi_b + u_3 = l_b$. Hence, our assumption that $\cos(t\pi/4) < l_b/l_a \leq 1$ implies that the upper bound on firm $a$’s payoff is sharp and coincides with the maximal payoff it can obtain. In fact, this also shows that $\pi^\text{max}_b = l_b$ is a sharp bound for firm $b$’s maximal payoff.

**Cases in which $(q_a, q_b) = (3, 2)$** We consider next how the bounds work when $(q_a, q_b) = (3, 2)$, in which case $v(N) = l_a + l_b + l_a \cos(t\pi/8)$. The conditions that $(u_1, u_2, u_3, \pi_a, \pi_b)$ must satisfy are:

$$0 \leq \pi_a \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8)$$

$$0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4)$$

$$0 \leq \pi_a + \pi_b \leq l_a + l_b + l_a \cos(t\pi/8)$$

$$l_a \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_a \cos(t\pi/8)$$

$$l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_a \cos(t\pi/8)$$

$$l_b \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)$$
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a + l_b \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

\[ l_a \leq \pi_a + u_1 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) \leq \pi_a + u_2 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a \cos(t\pi/4) \leq \pi_a + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) - l_b \cos(t\pi/4) \]

\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \]
\[ l_a + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a + l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

\[ l_b \cos(t\pi/4) \leq \pi_b + u_1 \leq l_a + l_b - l_a \cos(t\pi/4) \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_3 \leq l_b \]

\[ l_b \cos(t\pi/8) \leq \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_3 \leq l_a + l_b \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

\[ 0 \leq u_1 \leq l_a \]
\[ 0 \leq u_2 \leq l_a \cos(t\pi/8) \]
\[ 0 \leq u_3 \leq l_b \]
The unique nontrivial partition given \((q_a, q_b) = (3, 2)\) is \(P^*_3 = \{(a, 1, 2), \{b, 3\}\}\). Under \(P^*_3,\) we now show that we get sharp bounds for \(\pi^\text{min}_a, \pi^\text{min}_b, \pi^\text{max}_a\) and \(\pi^\text{max}_b\) respectively using \(w^*_a = 0, \ w^*_b = 0, \ w^*_a = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)),\) and \(w^*_b = l_b - l_a \cos(t\pi/4)\).

Thus, using the inequalities that define the core of this game, it can be shown that \(\pi^\text{min}_a = 0\) can arise as a feasible core allocation for firm \(a\). The core payoffs for the other players are \(u_1 = l_a, \ u_2 = l_a \cos(t\pi/8),\) and any pair of values \((\pi_b, u_3)\) such that \(0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4), \ l_a \cos(t\pi/4) \leq u_3 \leq l_b\) and \(\pi_b + u_3 = l_b\) hold. Hence, our assumption that \(\cos(t\pi/4) < l_b/l_a \leq 1\) implies that the lower bound on firm \(a\)'s payoff is sharp and coincides with the minimal payoff it can obtain. Indeed, this also proves that \(\pi^\text{min}_b = 0\) is a sharp bound for firm \(b\)'s minimal payoff, whereas \(\pi^\text{max}_b = l_b - l_a \cos(t\pi/4)\) is a sharp bound for firm \(b\)'s maximal payoff.

Using the inequalities that define the core of this game, it can also be shown that \(\pi^\text{max}_a = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\) can arise as a feasible core allocation for firm \(a\). The core payoffs for the other players are \(u_1 = l_b \cos(t\pi/4), \ u_2 = l_b \cos(t\pi/8),\) and any tuple of values \((\pi_b, u_3)\) such that \(0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4), \ l_a \cos(t\pi/4) \leq u_3 \leq l_b,\) and \(\pi_b + u_3 = l_b\) hold. Hence, our assumption that \(\cos(t\pi/4) < l_b/l_a \leq 1\) implies that the upper bound on firm \(a\)'s payoff is sharp and coincides with the maximal payoff it can obtain.

**Cases in which \((q_a, q_b) = (2, 3)\)**

We turn now to the situations in which \((q_a, q_b) = (2, 3)\). Given that \(v(N) = l_a + l_b + l_a \cos(t\pi/8)\), the conditions that \((u_1, u_2, u_3, \pi_a, \pi_b)\) must satisfy are:

\[
0 \leq \pi_a \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) - l_b \cos(t\pi/4)
\]

\[
0 \leq \pi_b \leq l_b
\]

\[
0 \leq \pi_a + \pi_b \leq l_a + l_b + l_a \cos(t\pi/8)
\]
\[ l_a \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_b \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a + l_b \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a \leq \pi_a + u_1 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) \leq \pi_a + u_2 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a \cos(t\pi/8) \leq \pi_a + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \]
\[ l_a + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) \]
\[ l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/4) \]
\[ l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[ l_a \cos(t\pi/4) \leq \pi_b + u_1 \leq l_a + l_b - l_a \cos(t\pi/4) \]
\[ l_a \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b \leq \pi_b + u_3 \leq l_b \]
\[ l_b \cos(t\pi/4) + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_2 \leq l_b + l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_3 \leq l_a + l_b \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]
\[
\begin{align*}
0 & \leq u_1 \leq l_a \\
0 & \leq u_2 \leq l_a \cos(t\pi/8) \\
0 & \leq u_3 \leq l_b \\
0 & \leq u_1 + u_2 \leq l_a + l_a \cos(t\pi/8) \\
0 & \leq u_1 + u_3 \leq l_a + l_b \\
0 & \leq u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \\
0 & \leq u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\end{align*}
\]

The unique nontrivial partition given \((q_a, q_b) = (2, 3)\) is \(P_{2,3}^* = \{\{a, 1, 2\}, \{b, 3\}\}\). Under \(P_{2,3}^*\), we now show that we get sharp bounds for \(\pi_a^{\min}, \pi_b^{\min}, \pi_a^{\max}\) and \(\pi_b^{\max}\) respectively using \(w_a^* = 0, w_b^* = 0, w_{-a}^* = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\), and \(s_{-b}^* = l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8)\) (note that the bound given by firm \(b\)'s weak competitive intensity for buyer 3 does not work well, so it needs to be strengthened).

Thus, using the inequalities that define the core of this game, it can be shown that \(\pi_a^{\min} = 0\) can arise as a feasible core allocation for firm \(a\). The core payoffs for the other players are \(u_1 = l_a, u_2 = l_a \cos(t\pi/8)\), and any pair of values \((\pi_b, u_3)\) such that \(0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4), l_a \cos(t\pi/4) \leq u_3 \leq l_b\) and \(\pi_b + u_3 = l_b\) hold. Hence, our assumption that \(\cos(t\pi/4) < l_b/l_a \leq 1\) implies that the lower bound on firm \(a\)'s payoff is sharp and coincides with the minimal payoff it can obtain. Indeed, this also proves that \(\pi_b^{\min} = 0\) is a sharp bound for firm \(b\)'s minimal payoff.

Using the inequalities that define the core of this game, it can also be shown that \(\pi_a^{\max} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\) can arise as a feasible core allocation for firm \(a\).\(^{20}\) The core payoffs for the other players are \(u_1 = l_b \cos(t\pi/4), u_2 = l_b \cos(t\pi/8)\), and any tuple of values \((\pi_b, u_3)\) such that \(0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8), l_a \cos(t\pi/4) - (l_a - l_b) \cos(t\pi/8) \leq u_3 \leq l_b,\) and \(\pi_b + u_3 = l_b\) hold. Hence, our assumption that \(\cos(t\pi/4) < l_b/l_a \leq 1\) implies that the upper bound on firm \(a\)'s payoff is sharp and coincides with the maximal payoff it can obtain. Indeed, this also proves that \(\pi_b^{\max} = l_b - l_a \cos(t\pi/4) + (l_a - l_b) \cos(t\pi/8)\) is a sharp bound for firm \(b\)'s maximal

\(^{20}\)In verifying that \(\pi_a^{\max} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))\) satisfies the constraints required by the core, we have used the fact that \(l_a \cos(t\pi/4) + l_b \cos(t\pi/8) - l_a \cos(t\pi/8) > 0\), which was shown to always hold (see expression (20)).
payoff.

**Cases in which** \((q_a, q_b) = (3, 3)\)  
We consider finally cases in which \((q_a, q_b) = (3, 3)\). Taking into account that \(v(N) = l_a + l_b + l_a \cos(t\pi/8)\), the conditions that \((u_1, u_2, u_3, \pi_a, \pi_b)\) must satisfy are:

\[
0 \leq \pi_a \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8)
\]
\[
0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4)
\]
\[
0 \leq \pi_a + \pi_b \leq l_a + l_b + l_a \cos(t\pi/8)
\]

\[
l_a \leq \pi_a + \pi_b + u_1 \leq l_a + l_b + l_a \cos(t\pi/8)
\]
\[
l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 \leq l_a + l_b + l_a \cos(t\pi/8)
\]
\[
l_b \leq \pi_a + \pi_b + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\]

\[
l_a + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8)
\]
\[
l_a + l_b \leq \pi_a + \pi_b + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\]
\[
l_b + l_a \cos(t\pi/8) \leq \pi_a + \pi_b + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\]

\[
l_a \leq \pi_a + u_1 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/8)
\]
\[
l_a \cos(t\pi/8) \leq \pi_a + u_2 \leq l_a + l_a \cos(t\pi/8) - l_b \cos(t\pi/4)
\]
\[
l_a \cos(t\pi/4) \leq \pi_a + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8) - l_b \cos(t\pi/4)
\]

\[
l_a + l_a \cos(t\pi/8) \leq \pi_a + u_1 + u_2 \leq l_a + l_a \cos(t\pi/8)
\]
\[
l_a + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/8)
\]
\[
l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) - l_b \cos(t\pi/4)
\]
\[
l_a + l_a \cos(t\pi/8) + l_a \cos(t\pi/4) \leq \pi_a + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8)
\]

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\[ l_b \cos(t\pi/4) \leq \pi_b + u_1 \leq l_a + l_b - l_a \cos(t\pi/4) \]
\[ l_b \cos(t\pi/8) \leq \pi_b + u_2 \leq l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b \leq \pi_b + u_3 \leq l_b \]
\[ l_b \cos(t\pi/8) + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_2 \leq l_a + l_b + l_a \cos(t\pi/8) - l_a \cos(t\pi/4) \]
\[ l_b + l_b \cos(t\pi/4) \leq \pi_b + u_1 + u_3 \leq l_a + l_b \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_2 + u_3 \leq l_b + l_a \cos(t\pi/8) \]
\[ l_b + l_b \cos(t\pi/8) \leq \pi_b + u_1 + u_2 + u_3 \leq l_a + l_b + l_a \cos(t\pi/8) \]

The unique nontrivial partition given \((q_a, q_b) = (3, 3)\) is \(P_{3,3}^* = \{\{a, 1, 2\}, \{b, 3\}\}\). Under \(P_{3,3}^*\), we proceed to show that we get sharp bounds for \(\pi_{a_{\min}}^*, \pi_{b_{\min}}^*, \pi_{a_{\max}}^*\) and \(\pi_{b_{\max}}^*\) respectively using 
\[ w_a^* = 0, \quad w_b^* = 0, \quad w_{a_{-a}}^* = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4)), \quad w_{-b}^* = l_b - l_a \cos(t\pi/4). \]

Thus, using the inequalities that define the core of this game, it can be shown that \(\pi_{a_{\min}}^* = 0\) can arise as a feasible core allocation for firm \(a\). The core payoffs for the other players are 
\[ u_1 = l_a, \quad u_2 = l_a \cos(t\pi/8), \quad u_3 = l_a \cos(t\pi/8) \] and any pair of values \((\pi_b, u_3)\) such that \(0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4)\), 
\[ l_a \cos(t\pi/4) \leq u_3 \leq l_b \] and \(\pi_b + u_3 = l_b\) hold. Hence, our assumption that \(\cos(t\pi/4) < l_b/l_a \leq 1\) implies that the lower bound on firm \(a\)'s payoff is sharp and coincides with the minimal payoff it can obtain. Indeed, this also proves that \(\pi_{b_{\min}}^* = 0\) is a sharp bound for firm \(b\)'s minimal payoff,
whereas $\pi_b^{\text{max}} = l_b - l_a \cos(t\pi/4)$ is a sharp bound for firm $b$’s maximal payoff.

Using the inequalities that define the core of this game, it can also be shown that $\pi_a^{\text{max}} = l_a(1 + \cos(t\pi/8)) - l_b(\cos(t\pi/8) + \cos(t\pi/4))$ can arise as a feasible core allocation for firm $a$. The core payoffs for the other players are $u_1 = l_b \cos(t\pi/4)$, $u_2 = l_b \cos(t\pi/8)$, and any tuple of values $(\pi_b, u_3)$ such that $0 \leq \pi_b \leq l_b - l_a \cos(t\pi/4)$, $l_a \cos(t\pi/4) \leq u_3 \leq l_b$, and $\pi_b + u_3 = l_b$. Hence, our assumption that $\cos(t\pi/4) < l_b/l_a \leq 1$ implies that the upper bound on firm $a$’s payoff is sharp and coincides with the maximal payoff it can obtain.