

# Asset Pricing Implications of Strategic Trading and Activism\*

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## Abstract

This paper studies the asset pricing implications of investor activism. We consider a setting where a blockholder can both trade the firm's shares and influence the firm value by exerting costly effort. The blockholder's ability to add value varies over time.

To understand the effects of blockholder transparency, we consider two regimes: one in which the market observes the blockholder's ability and another where it does not. *i*) When the market observes the blockholder's ability, the blockholder's trading is characterized by Coasian dynamics: despite his large size, the blockholder is unable to exploit his market power and trades at a competitive price. Furthermore, when his ability to add value increases, the blockholder sells shares which in turn diminishes his incentives to boost the firm's productivity. *ii*) By contrast, when the blockholder privately observes his ability, the dynamics of trading and activism change drastically: as the blockholder's ability improves, he buys shares. He does so gradually to mitigate the price impact of his trading. As the blockholder's stock-holdings grow, a virtuous circle unravels that boosts the firm's productivity and reduces the firm's cost of capital.

In the long-run, the presence of information asymmetry about blockholder ability leads to a more concentrated ownership, more intense activism, and higher firm productivity. However, it also leads to more volatile cash flows, and a higher risk-premium. The implications for the firm's stock price are ambiguous.

**Keywords: Strategic Trading, Activism, Asset Pricing.**

**JEL Classification:** D72, D82, D83, G20.

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# 1 Introduction

Activists have been around since the 1980s, but the scale of their operations in America is unprecedented. They run funds with nearly \$166 billion of capital, and in 2014 they attracted 20% of all hedge funds flows.<sup>1</sup> In the past five years, one in two firms in the S&P 500 index have had a big activist fund on its share register, and one in seven has been attacked by an activist. As the Economist puts it “the only proven defense that a firm can offer is to not be American in the first place; 80% of activist interventions are in America, where the culture and legal system is better suited to shareholder revolts than those in Europe or Asia.”

In theory, activists play a governance role that benefits public companies, since most investors take little interest in how firms are managed. However, in practice firms face significant uncertainty whether a prospective activist will create value (see [Cronqvist and Fahlenbrach \(2008\)](#)).<sup>2</sup> Indeed, a blockholder knows better than anyone what his incentives are: the extent to which he is willing to bear the cost of activism, his rent-seeking motives, and the intended horizon of his interventions. In this paper, we study the impact of information asymmetry on the dynamics of activism and its asset pricing consequences.

We study a dynamic game between a large investor (henceforth, blockholder) and a competitive fringe of investors (henceforth, the market). There is a single firm and a risk-free asset. At each point the blockholder can both trade and influence the firm’s productivity via costly effort. The blockholder cannot commit to hold a large block in the future and trades freely based on his information and hedging needs. Effort is unobservable and the market only observes the firm’s cash flows and the blockholder’s stake. Furthermore, the market is uncertain about the blockholder’s motives. In particular, we assume there is information asymmetry about the blockholder’s ability to add value; only the blockholder observes his ability. Also, since in practice a blockholder’s circumstances change, we assume the blockholder’s ability varies in a random but persistent fashion. In our baseline setting, this is the only source of asymmetric information. Hence, a separating equilibrium is fully revealing, and the blockholder’s trading choice is affected by signaling incentives. We later extend the model and consider the case when the blockholder is subject to unobservable private benefits (or, liquidity shocks). In this case, the equilibrium is no longer fully revealing and the market must infer whether the blockholder is selling due to low ability or liquidity shocks. Interesting dynamics arise as the effort choice of the blockholder is distorted by reputation concerns.

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<sup>1</sup>See “Shareholder Activist: Capitalism Unlikely Heroes,” The Economist, February, 2015.

<sup>2</sup>[Dou et al. \(2016\)](#) note that blockholders have heterogeneous beliefs, skills, and preferences and influence corporate policies and operational decisions through different channels and to various extents. These channels include direct communication with management, insider positions (management or director), and changes to corporate governance practices such as board characteristics.

A blockholder who is selling has incentives to increase performance by increasing effort to rise the price while a blockholder who is buying has incentives to depress performance by reducing effort. This is the traditional ratchet effect in career concern models. However, unlike in traditional models of the ratchet effect, the direction of the distortion is endogenous and depends on the trading strategy of the blockholder.

The market has a limited risk-bearing capacity and sets the stock price competitively based on the blockholder's trading history and the expected evolution of the firm cash flows. Since the market does not observe blockholder ability, in valuing the firm shares the market must assess blockholder ability based on the blockholder's *skin in the game*, as captured by the blockholder stock-holdings (and trading history). This means the blockholder faces a relatively illiquid market because every time he trades the market revises its belief about ability and updates the stock price accordingly. Given the price impact of his trading, the blockholder must trade gradually to retain as much as possible the surplus he expects to generate by boosting the firm's productivity via effort.

The blockholder affects the firm value through three channels which we label productivity, volatility, and risk-sharing channels. First, the blockholder directly affects the firm's productivity by exerting effort. Second, the blockholder affects the volatility of the firm cash-flows because his ability is subject to shocks. This causes variation in the blockholder's effort and stock-holdings, which in turn causes variation in the firm cash-flows. This channel by itself affects the firm risk-premium (i.e., cost of capital) because it makes the business more risky, particularly when the blockholder ability is unobservable. The risk premium is also affected through a third channel: the market has limited risk-bearing capacity and demands a higher risk premium when the blockholder sells shares, because the market absorbs a greater fraction of the firm's cash-flow risk.<sup>3</sup>

We find that the presence of asymmetric information (i.e., unknown blockholder ability) drastically changes the firm's ownership structure, the long-run distribution of cash flows, and the dynamics of activism and asset prices. Though the presence of information asymmetry reduces the liquidity facing the blockholder, it concentrates the firm's ownership thereby boosting the blockholder's activism. In the long-run, the blockholder retains a larger stake, effectively holding an undiversified portfolio, unlike in the case with observable ability. This lack of diversification caused by information asymmetry, enhances the blockholder's activism incentives, thus mitigating the free-rider problem that undermines the firm's productivity. We thus find that opacity about the blockholder's incentives may strengthen the productivity benefits of investor activism.

To understand how asymmetric information affects the dynamics of activism, we consider — as a benchmark — the situation where the market observes blockholder ability. In this case, a positive

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<sup>3</sup>This channel is likely to be less relevant for activists who hold 5% of the firm shares, but more relevant for other types of blockholders who hold larger blocks of 20% or more.

shock to blockholder ability causes an immediate increase in the stock price, in anticipation of the firm realizing a higher productivity due to more effective and intense activism, other things equal. Because the activist is more efficient, a smaller block is needed to provide the same incentives to exert effort. In addition, a risk averse blockholder would like to hedge the shocks to his productivity. For these reasons, the blockholder trades against their productivity shocks, which means that the blockholder sells after positive productivity shocks. By reducing his holdings, the blockholder essentially weakens his incentives to monitor the firm (thus exacerbating the free-rider problem) and deprives other shareholders from the potential benefits of activism. In this context, the blockholder's trading is characterized by Coasian dynamics. The blockholder has market power but cannot exploit it given his lack of commitment.<sup>4</sup> Such a commitment problem in models with large shareholders has already been studied by Kihlstrom (2000), DeMarzo and Urošević (2006) and Gorton et al. (2014). We contribute to this literature by incorporating the effect of asymmetric information.

We show that asymmetric information qualitatively changes the trading dynamics. Initially, when a positive shock hits, the blockholder buys shares (given the initial underpricing of the stock) and holds them while the shock persists. The Coase conjecture no longer holds because of signaling effects. The blockholder faces an illiquid market in which his trading has a significant price impact. Such illiquidity provides a commitment device. When the market observes that the blockholder is buying shares, it updates its beliefs about the blockholder's ability and the price jumps. This price impact effectively moderates the speed at which the blockholder trades; the blockholder trades slowly, particularly when his ability is persistent. As the blockholder's stock ownership grows, two effects unfold: the firm's risk premium decreases, because the market absorbs a smaller fraction of the firm's risk, and the firm's productivity improves, because the blockholder's monitoring incentives become stronger.

Even though shocks are transitory, we show that the unobservable nature of blockholder ability not only alters the dynamics of trading and effort (relative to the observable case) but also has significant long-run consequences. In steady state, the blockholder's stock-holdings, the mean and variance of cash flows, as well as the risk premium and return distribution change significantly, relative to a situation in which blockholder ability is observable. The unobservable case generates a more concentrated ownership structure, with the blockholder holding a larger stake. This mitigates the free-riding problem: the intensity of activism goes up and the firm's productivity grows. On the downside, cash flows become significantly more volatile.

For a fixed distribution of cash flows, the larger ownership of the blockholder caused by the

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<sup>4</sup>This lack of commitment was first studied by Coase (1972)). The paradox asserts that a monopolist selling durable goods (e.g., houses) effectively competes against his future sales. Anticipating this form of competition, the monopolist would choose to charge a competitive price in the first place. The monopolist's inability to commit to not selling all his goods, so to exploit his market power, would eliminate his monopoly rents, in a dynamic context.

information asymmetry, generates pressure to lower the risk premium, as the market absorbs a smaller fraction of the firm risk. But there is another effect running in the opposite direction. Information asymmetry increases cash-flow volatility. The cash flow becomes more sensitive to variation in the blockholder's ability, hence more volatile. The risk-premium is affected by these two countervailing forces: risk-sharing and cash flow volatility. Our numerical analysis shows that the volatility effect dominates leading to a higher risk premium, despite the blockholder absorbing a larger fraction of the risk, relative to the observable case.

In sum, under unobservable ability, the firm becomes more productive but also more risky. As a result, the effect of unobservable ability on the average stock price is ambiguous. When the blockholder is very risk averse and holds a small stake, the higher risk-premium effect dominates the favorable productivity benefits. By contrast, when the blockholder is not too risk averse and is willing to hold a relatively large stake, the productivity benefits dominate the higher risk premium, leading to a higher stock price in the long-run.

**Literature** The most closely related papers are [Huddart \(1993\)](#), [Admati et al. \(1994\)](#) and [DeMarzo and Urošević \(2006\)](#), who study the incentives of large shareholders to monitor. They emphasize the blockholder's lack of commitment and free riding problem, and highlight the tension between optimal risk diversification and monitoring incentives, which requires concentrated ownership. Even though large shareholders add value due to their incentives to monitor the firm, [Admati et al. \(1994\)](#) and [DeMarzo and Urošević \(2006\)](#) show that these large blocks are unstable because an undiversified blockholder would like to reduce his risk exposure by selling shares over time. One policy implication coming out of these models is that corporate governance could be improved if blockholders are subsidized to hold large blocks.

In contrast, [Leland and Pyle \(1977\)](#) shows that, in the presence of asymmetric information, a risk averse entrepreneur will retain ownership to signal that the value of the firm is high. A natural implication is that asymmetric information might alleviate the commitment problem identified in [Admati et al. \(1994\)](#) and [DeMarzo and Urošević \(2006\)](#), and increase value when cash-flows are endogenous and there is moral hazard. Our setting is based on [DeMarzo and Urošević \(2006\)](#). Our main contribution, relative to them is to allow for information asymmetry regarding the blockholder's ability to add value. [Gomes \(2000\)](#) also studies a reputation game, with two types of manager/owners, who differ in terms of their cost of effort. In [Gomes \(2000\)](#) the manager effort is observable, and he shows how reputation effects moderate the insider's incentive to expropriate minority shareholders. Unlike [Gomes](#), we allow for hidden effort and time-varying ability. Moreover, our main focus is not the effect of reputation on managerial incentives but rather to show how price impact due to asymmetric information can reduce the commitment problem referenced

above and its asset pricing implications.

More broadly, our paper belongs to the large literature in corporate governance that looks at the effect of blockholders and activist investors. This literature is surveyed in [Becht, Bolton, and Röell \(2003\)](#) and [Edmans and Holderness \(2017\)](#). Starting with [Hirschman \(1970\)](#), the literature on corporate governance has look at how investors can affect corporate decision by voice (direct intervention) or exit (showing their discontent by selling their shares).

[Admati and Pfleiderer \(2009\)](#) and [Edmans \(2009\)](#) show that investor can intervene in the corporation using by exiting their incentives when they disagree with the companies management. The key assumption in these models is that the manager’s compensation is tied to the price of the company, so the manager is hurt when selling pressures bring the price down.

Our paper belongs to the literature looking at the impact of investors in through direct intervention. A key issue in this literature is that, when the firm is underperforming, blockholder may have incentives to sell (cut and run) instead of bear the cost of intervening. For this reason, it has been suggested that market liquidity might harm corporate governance ([Coffee, 1991](#)). For example, motivated by this idea, the European Union agreed to implement a transaction tax in September 2016. This trade-off between governance and liquidity has been formally analyzed by [Kahn and Winton \(1998\)](#), [Noe \(2002\)](#) and [Faure-Grimaud and Gromb \(2004\)](#). A counterargument to thesis in ([Coffee, 1991](#)) is that liquidity might reduce the free riding problem identified by [Grossman and Hart \(1980\)](#) and [Shleifer and Vishny \(1986\)](#). By facilitating the creation of a large block in the first place, liquidity can actually improve corporate governance. These argument is formalized by [Kyle and Vila \(1991\)](#), in the contexts of takeovers, and [Maug \(1998\)](#) in the contexts of investor activism.

Most of these models are static in nature, which prevents to incorporate the effect of future trading identified by [Admati et al. \(1994\)](#) and [DeMarzo and Urošević \(2006\)](#), and to derive asset pricing implications of investor activism. Recently, [Back et al. \(2018\)](#) have analyzed many of these issues in a fully dynamic setting. They consider a setting similar to [Kyle \(1985\)](#) in which an informed trader has private information about his initial stock holdings, and can exert costly effort to increase the firm value before it becomes known. Surprisingly, and in contrast to [Kyle \(1985\)](#), they find that the relation between efficiency and liquidity is ambiguous and crucially depends on model parameters. Because liquidity and intervention are simultaneously determined, more noise trading can increase the information asymmetry about the activist’s intentions and thus decrease liquidity. Unlike [Back et al. \(2018\)](#), we consider a setting in which intervention is continuous (rather than at the end), the block size is observable, and there is asymmetric information about the activist productivity. We also consider a setting with risk-averse market makers and activists, which introduce a trade-off between activism, which requires large blocks, and diversification. Moreover, our setting with risk aversion allow us to explore the asset pricing implications of activism, which

is one of the main purposes of our paper.

Finally, there is a relatively small literature in asset pricing that looks at the asset pricing implications of agency frictions in general equilibrium settings. The main lesson from this literature is that, by distorting productive decisions, agency frictions affect the volatility of cash-flows and the overall risk premium. For example, [Gorton et al. \(2014\)](#) considers a Lucas-tree economy, in which the output is determined by the effort of a manager whose compensation depends on output and who can trade the shares of the asset. They show that depending on the risk aversion of the manager, trading by the manager can lead to more or less volatile cash flows and a risk premium. [Albuquerque and Wang \(2008\)](#) study the effect of investor protection on welfare and asset pricing in a general equilibrium model with production. They show that weaker investor protection increases agency costs, which lead to over-investment, more volatile cash-flows and a larger risk premium.

## 2 Model

We study the behavior of a large investor (henceforth, blockholder) who can both trade the firm's stock and undertake costly actions to affect the firm's output.

In addition, there is a continuum of competitive investors who also trade shares but do not influence the firm. All agents in the economy maximize expected utility and have CARA preferences. Hence, as [DeMarzo and Urošević \(2006\)](#) we can aggregate the competitive investors into a single, aggregate investor with risk aversion parameter  $\gamma_M$ .

Time  $t$  is continuous and the horizon is infinite. There is a single firm in unit supply with a cumulative cash flow process  $(D_t)_{t \geq 0}$  evolving as

$$dD_t = (\mu_D + a_t)dt + \sigma_D dB_t^D,$$

where  $a_t$  is the blockholder's effort and  $(B_t^D)_{t \geq 0}$  is a standard Brownian motion. The cash flow  $dD_t$  is publicly observable but  $a_t$  is not. The market thus faces a moral hazard problem. Without loss of generality we assume that the realized cash flows are paid to shareholders each period, and interpret  $dD_t$  as the firm dividends.

The blockholder's effort affects the firm cash flows. This captures the notion that the blockholder's effort has an externality on the firm's productivity. When  $a_t > 0$  the externality is positive. We allow  $a_t < 0$ , in which case  $a_t$  represents the blockholder's rent extraction. We are agnostic as to the source of the externality. As in [Admati et al. \(1994\)](#) we can think of  $a_t$  as the blockholder's monitoring effort—which disciplines managers and mitigates agency conflicts—or as the influence the blockholder exerts on the firm's management. Examples of  $a_t$  include public criticism of man-

agement or launching a proxy fight, advising management on strategy, figuring out how to vote on proxy contest launched by others or not taking private benefits for himself.

Effort is costly and the cost is borne entirely by the blockholder.<sup>5</sup> In that sense, the market free rides on the blockholder’s effort. The blockholder’s cost of effort depends on his ability  $(\zeta_t)_{t \geq 0}$ . Specifically, the cost of effort is given by

$$\Phi(a_t, \zeta_t) = \phi a_t^2 - \psi \zeta_t a_t,$$

Hence, the cost of effort depends on two variables: blockholder effort  $a_t$ , and blockholder ability  $\zeta_t$ .<sup>6</sup> Broadly, the term  $\psi \zeta_t a_t$  captures private benefits that the blockholder receives from his effort to influence the firm. Cross sectional differences in ability and preferences are realistic: [Cronqvist and Fahlenbrach \(2008\)](#) find significant blockholder fixed effects in investment, financial, and executive compensation policies.<sup>7</sup>

The blockholder privately observes his ability  $(\zeta_t)_{t \geq 0}$ . Ability is random but persistent. In particular, it evolves according to a mean reverting process

$$d\zeta_t = -\kappa \zeta_t dt + \sigma_\zeta dB_t^\zeta,$$

where  $(B_t^\zeta)_{t \geq 0}$  is a Brownian motion independent of  $(B_t^D)_{t \geq 0}$ . The speed of mean reversion is thus captured by  $\kappa$ . When  $\kappa$  is small, ability shocks are highly persistent. In particular,  $(\zeta_t)_{t \geq 0}$  is a stationary Gaussian process

$$\mathbb{E}[\zeta_t] = 0 \text{ and } \text{Cov}[\zeta_t, \zeta_s] = \frac{\sigma_\zeta^2}{2\kappa} e^{-\kappa|t-s|}.$$

The variance of the stationary distribution of  $\zeta_t$  is given by  $\bar{\sigma}_\zeta^2 \equiv \sigma_\zeta^2/2\kappa$ .

All agents are risk averse and have preferences with constant absolute risk aversion. Specifically, the flow utility of a trader type  $i$  is represented by CARA utility function

$$u_i(c) = -\exp(-\gamma_i c)$$

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<sup>5</sup>Bill Ackman, a well known hedge fund activist, asserts “Shareholder activism is extremely time-consuming, expensive and a drain on an investment firm’s resources.” See “For Activist Investors, a Wide Reporting Window”, The New York Times, May 19, 2014.

<sup>6</sup>It is natural to think that the blockholder’s ability to influence the firm depends on its holdings,  $X$ . The model does not qualitatively change if the cost function includes a term  $-\chi a X$ , but to simplify the exposition we don’t include it.

<sup>7</sup>The Economist analyzed the 50 largest activist positions in America since 2009 and found that on average, profits, capital investment, and R&D have risen. See “Shareholder activism Capitalism’s unlikely heroes”, The Economist, February, 2015.



for  $i \in \{L, M\}$  where  $c$  is consumption and  $\gamma_i$  is the coefficient of risk aversion of a type  $i$  investor. In this context  $\gamma_L/\gamma_M$  represents the market's risk-bearing capacity.

The information structure is as follows. The blockholder observes the dividend  $dD_t$  and his ability  $\zeta_t$ . Based on this information set, the blockholder chooses consumption/savings  $c_t$  and stock holdings  $X_t$ , where  $X_t$  is the number of shares the blockholder holds at time  $t$ .

Competitive investors observe the dividend process  $D_t$  as well as the large shareholder's order flow  $q_t^L$ . Hence, the competitive investors information set is given by the filtration  $\mathcal{F}_t^M = \sigma(D_s, q_s^L | s \leq t)$ , while the blockholder's information set is given by the filtration  $\mathcal{F}_t^L = \sigma(D_s, q_s^L, \zeta_s | s \leq t)$ . Throughout the paper, we use the notation  $\mathbb{E}_t^M[\cdot] \equiv \mathbb{E}[\cdot | \mathcal{F}_t^M]$  and  $\mathbb{E}_t^L[\cdot] \equiv \mathbb{E}[\cdot | \mathcal{F}_t^L]$ , and denote  $\hat{\zeta}_t \equiv \mathbb{E}^M[\zeta_t]$ .

Competitive investors choose a consumption  $c_t^M$  and order flow  $q_t^M$  strategy adapted to  $\mathcal{F}_t^M$ . We denote the aggregate holdings of market makers at time  $t$  by  $Y_t$ . Since the firm is in unit supply the market clearing condition at time  $t$  is

$$X_t + Y_t = 1,$$

Hence, the holdings  $X_t$  and  $Y_t$ , represent the shareholder and competitive investors percentage of ownership, respectively. We follow [Kyle, Obizhaeva, and Wang \(2017\)](#) and consider equilibria with smooth trading in which the blockholder inventory  $X_t$  is an absolutely continuous process, so the market clearing condition requires that at any time  $t$

$$q_t^M + q_t^L = 0.$$

**Optimization Program** Denote by  $W_t$  the savings of a market maker. Given a  $\mathcal{F}_t^M$ -adapted price process  $p_t$ , at any time  $t$ , the competitive investor chooses a  $\mathcal{F}_t^M$ -adapted strategy  $(c_t^M, q_t^M)_{t \geq 0}$  to solve the following problem

$$\max_{c, q^M} \mathbb{E}_t^M \left[ \int_0^\infty e^{-r(s-t)} u_M(c_s) ds \right]$$

subject to

$$\begin{aligned} dW_t &= (rW_t - c_t - p_t q_t^M + (\mu_D + a_t)Y_t)dt + \sigma_D Y_t dB_t^D \\ dY_t &= q_t^M dt. \end{aligned}$$

The second equation captures the market maker's budget constraint. The market maker's savings account grows at the interest rate  $r$ . The market makers consumes  $c_t$  invests  $p_t q_t$  in additional shares and receives  $dD_t$  as dividends on their existing shares.

Observe, that because market makers are a competitive fringe they take the price  $p_t$  as given; in other words their order flow does not have a price impact. On the other hand, the blockholder chooses a  $\mathcal{F}_t^L$ -adapted strategy  $(c_t^L, q_t^M, a_t)_{t \geq 0}$  to solve the following problem

$$\begin{aligned} & \max_{c, q^L, a} \mathbb{E}_t^L \left[ \int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\ & \text{subject to} \\ & dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - p_t q_t^L + (\mu_D + a_t) X_t) dt + X_t \sigma_D dB_t^D \\ & dX_t = q_t^L dt. \end{aligned}$$

The blockholder chooses effort  $a_t$ , consumption  $c_t$ , and an order flow  $q_t^L$ . The blockholder is privately informed about  $\zeta_t$  so, unlike the market, he does not need to form beliefs about  $\zeta_t$ . Also, the blockholder has market power, hence he takes into consideration the price impact of his order flow  $q_t^L$ . In fact, his order flow affects the price for two reasons: because of competition and because it conveys information about his ability  $\zeta_t$ .

In summary, two things differentiate the problem of the blockholder from that of market makers. First, the blockholder does not take the price as given. Second, the blockholder bears the cost of effort  $\Phi(a_t, \zeta_t)$  (More generally, we can think of  $\Phi(a_t, \zeta_t)$  as capturing the cost of effort net of the blockholder's private benefits.)

**Equilibrium definition** An equilibrium is a price process  $p_t$  and a profile  $(q_t^L, q_t^M, a_t)$  such that  $q_t^M$  solves the market makers' portfolio problem,  $(q_t^L, a_t)$  solves the large shareholder's problem, and the market clearing condition  $q_t^L = -q_t^M$  is satisfied.

We consider stationary Markov perfect equilibria in which  $(p_t, q_t^L, q_t^M, a_t)$  are affine functions of the three state variables  $(X_t, \zeta_t, \hat{\zeta}_t)$  where

$$\begin{aligned} q_t^L &= Q_0 - Q_x X_t + Q_\zeta \zeta_t \\ a_t &= A_x X_t + A_\zeta \zeta_t \\ p_t &= P_0 + P_x X_t + P_\zeta \hat{\zeta}_t. \end{aligned}$$

Throughout the paper we use boldface to denote the coefficient vectors  $(\mathbf{Q}, \mathbf{A}, \mathbf{P})$ .

### 3 Competitive Investors' Problem

Market makers choose their portfolios based on their beliefs about the blockholder's ability  $\zeta_t$  and his trading strategy. In particular, given the conjectured strategy, and the blockholder's inventory  $X_t$ , the market makers can invert the order flow of the blockholder  $q_L$  to infer the exact value of the ability  $\zeta_t$ . Hence, the evolution of the market makers' belief is given by

$$d\hat{\zeta}_t = -\kappa\hat{\zeta}_t dt + \sigma_\zeta dB_t^\zeta.$$

As usual in Gaussian linear quadratic models with CARA preferences (e.g., see [Vayanos and Woolley \(2013\)](#)), we conjecture and then verify that the value function takes the form

$$J(W, Y, X, \hat{\zeta}) = -\exp\left(-r\gamma_M(W + H(Y, X, \hat{\zeta}))\right)/r,$$

where the function  $H$  is the certainty equivalent of a market maker and satisfies the following HJB equation:

$$\begin{aligned} rH = \max_q (\mu_D + A_x X + A_\zeta \hat{\zeta})Y - p(X, \hat{\zeta})q - \frac{1}{2}r\gamma_M (Y^2\sigma_D^2 + \sigma_\zeta^2 H_\zeta^2) \\ qH_Y + (Q_0 - Q_x X + Q_\zeta \hat{\zeta})H_X - \kappa\hat{\zeta}H_\zeta + \frac{1}{2}\sigma_\zeta^2 H_{\zeta\zeta} \end{aligned} \quad (1)$$

Taking the first order condition for  $q$  we get

$$p(X, \hat{\zeta}) = H_Y. \quad (2)$$

This condition states that for the market maker to be willing to trade, the price must equal the marginal impact of an additional share on the market maker's certainty equivalent, given his conjecture about  $\zeta_t$  and the strategy that the blockholder is expected to follow in the future. The market maker computes the firm value, given his belief  $\hat{\zeta}_t$  by projecting the trading strategy that the blockholder will adopt and the impact this will have on the firm's future dividends.

**Lemma 1.** *The market makers certainty equivalent is given by*

$$H(Y, X, \hat{\zeta}) = h_y Y + h_{yx} XY + h_{y\zeta} Y \hat{\zeta} + h_{yy} Y^2$$

such that

$$\begin{aligned}
h_y &= \frac{\mu_D}{r} + \frac{A_x Q_0}{r(r + Q_x)} \\
h_{yx} &= \frac{A_x}{r + Q_x} \\
h_{y\zeta} &= \frac{1}{r + \kappa} \left( A_\zeta + \frac{A_x Q_\zeta}{r + Q_x} \right) \\
h_{yy} &= -\frac{\gamma M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2).
\end{aligned}$$

The conjectured price function is

$$p(X, \hat{\zeta}) = P_0 + P_x X + P_\zeta \hat{\zeta}.$$

Since the market clearing condition,  $X + Y = 1$  we can match coefficients to arrive at the following equations:

$$P_0 = h_y + 2h_{yy} \quad (3a)$$

$$P_x = h_{yx} - 2h_{yy} \quad (3b)$$

$$P_\zeta = h_{y\zeta}. \quad (3c)$$

Observe, that these conditions apply not only to the case when  $\zeta_t$  is unobservable but also to that when the market observes ability  $\zeta_t$ .

Because of competition, the market maker breaks even for any order size the blockholder may place,  $q_L$ . The price is sensitive to the expected ownership of the blockholder  $(X_s)_{s \geq t}$ , for two reasons: first, the impact of the blockholder on the firm's productivity is linked to the blockholder's ownership. Second, the larger the blockholder stake, the lesser risk the market absorbs, which lowers the risk premium. We can build intuition about the determinants of the price by deriving an explicit representation of the stock price as the present value of future dividends minus the risk premium.

**Proposition 1.** *The equilibrium price satisfies*

$$p_t = \mathbb{E}_t^M \left[ \int_t^\infty e^{-r(s-t)} (\mu_D + A_x X_s + A_\zeta \zeta_s - r\gamma M \hat{\sigma}_D^2 (1 - X_s)) ds \right],$$

where

$$\hat{\sigma}_D^2 \equiv \sigma_D^2 + \frac{\sigma_\zeta^2}{(r + \kappa)^2} \left( A_\zeta + A_x \frac{Q_\zeta}{r + Q_x} \right),$$

The volatility of cash flow  $\hat{\sigma}_D^2$  captures the risk coming from the operations as well as the

risk coming from the blockholder's intervention. The shock to the blockholders' ability has an instantaneous effect, given by the coefficient effect  $A_\zeta$ , as well as a long-term effect given by the impact in the blockholder's position,  $A_x$ . The risk premium is given by  $r\gamma_M\hat{\sigma}_D^2(1 - X_s)$ .

The volatility of cash flows  $\hat{\sigma}_D^2$  has two components: First, a basic component that is independent of the blockholder ability and behavior,  $\sigma_D^2$ . The second component, depends on the volatility of the blockholder ability,  $\sigma_\zeta^2$ . This component is also affected by blockholder behavior. In particular, notice that the volatility impact of ability depends on how the blockholder trading responds to ability shocks, as captured by  $Q_\zeta$ . If  $Q_\zeta$  is negative then that means the blockholder effectively smooths the volatility impact of his ability shocks on the firm cash flows. Otherwise, the blockholder behavior has a multiplicative effect on cash flow volatility.

## 4 Benchmark: No Private Information

Before solving the blockholder's problem and characterizing the equilibrium in general, we study the case when  $\zeta_t$  is observable, using as a starting point the solution to the market makers' problem previously discussed. As a special case, we also present the solution when the ability of the blockholder is irrelevant  $\psi = 0$  which corresponds to the setting in [DeMarzo and Urošević \(2006\)](#).

When,  $\zeta_t$  is observable, the market no longer needs to form beliefs about  $\zeta_t$  and, for that reason, the price only depends on the holding  $X_t$  but it no longer depends on the blockholder's order flow,  $q_t^L$ . As before, we consider a linear equilibrium with the following structure:

$$\begin{aligned} q_t &= Q_0^o - Q_x^o X_t + Q_\zeta^{\text{obs}} \zeta_t \\ a_t &= A_x X_t + A_\zeta \zeta_t \\ p_t &= P_0^o + P_x^o X_t + P_\zeta^{\text{obs}} \zeta_t. \end{aligned}$$

The market makers' problem was already characterized in the previous section. The problem of the blockholder is

$$\begin{aligned} &\max_{c, q, a} \mathbb{E}_t^L \left[ \int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\ &\text{subject to} \\ &dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - P(X_t, \zeta_t)q_t + (\mu_D + a_t)X_t)dt + X_t\sigma_D dB_t^D \\ &dX_t = q_t dt. \end{aligned}$$

One can verify that the value function of the blockholder takes the form

$$V(W, X, \zeta) = -\exp(-r\gamma_L(W + G^o(X, \zeta))) / r,$$

where the certainty equivalent  $G$  satisfies the HJB:

$$rG^o = \max_{q,a} (\mu_D + a)X - \phi a^2 + \psi\zeta a - P(X, \zeta)q - \frac{1}{2}r\gamma_L(\sigma_D^2 X^2 + \sigma_\zeta^2 (G_\zeta^o)^2) + qG_X^o - \kappa\zeta G_\zeta^o + \frac{1}{2}\sigma_\zeta^2 G_{\zeta\zeta}^o \quad (4)$$

Taking the first order conditions, we get that

$$a = \frac{\psi\zeta + X}{2\phi} \quad (5a)$$

$$P(X, \zeta) = G_X^o \quad (5b)$$

Condition 5a states that the blockholder effort is a linear function of the blockholder's ability and holdings. This is intuitive: the blockholder exerts more effort when he is more productive. Also, since there is a free-riding problem, the blockholder exerts more effort when he owns more shares, as he then more fully internalizes the benefits of effort. Put differently, the free riding problem is milder when the blockholder's stake is larger.

Condition 5b says that the price must equal the marginal value of a share to the blockholder. Because of competition, the price also equals the marginal value to a market maker. Hence, when ability is observable, trading is characterized by Coasian dynamics. The blockholder trades all the way until the competitive price, despite having market power. Trade can be smooth, but at any point the blockholder effectively trades at a price that equals his marginal valuation, as predicted by the Coase conjecture.

As before, we conjecture and verify that the certainty equivalent is a quadratic function of  $X$  and  $\zeta$ , and obtain the following lemma, analogous to Lemma 3.

**Lemma 2.** *The large shareholder's certainty equivalent in the observable case is given by*

$$G^o(X, \zeta) = g_0^o + g_x^o X + g_{xx}^o X^2 + g_{\zeta\zeta}^o \zeta^2 + g_{x\zeta}^o X\zeta,$$

where

$$\begin{aligned}
g_0^o &= \frac{\sigma_\zeta^2}{r} g_{\zeta\zeta}^o \\
g_x^o &= \frac{\mu_D}{r} \\
g_{\zeta\zeta}^o &= \frac{\pm \sqrt{(r+2\kappa)^2 + 2r\gamma_L \sigma_\zeta^2 \frac{\psi^2}{\phi}} - (r+2\kappa)}{4r\gamma_L \sigma_\zeta^2} \\
g_{x\zeta}^o &= \frac{\psi}{2\phi(r+\kappa + 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^o)} \\
g_{xx}^o &= \frac{1}{4r\phi} - \frac{\gamma_L}{2} (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)
\end{aligned}$$

The maximal certainty equivalent corresponds to the positive root  $g_{\zeta\zeta}^o$ .

There are two solutions to this polynomial, which correspond to two different equilibria, but one of them dominates the other in terms of the blockholder's certainty equivalent.

The next step is to find expressions for the coefficients of the trading strategy,  $\mathbf{Q}$ . Since  $P = G_X^{\text{obs}}$ , we have that

$$P(X, \zeta) = g_x^o + 2g_{xx}^o X + g_{x\zeta}^o \zeta. \quad (6)$$

But from the solution of the market makers' problem we also have that

$$P(X, \zeta) = h_y^o + 2h_{yy}^o + (h_{yx}^o - 2h_{yy}^o)X + h_{y\zeta}^o \zeta. \quad (7)$$

where the coefficients  $\mathbf{h}^o$  are given by the same expressions as in Lemma 1. In equilibrium, both expressions must coincide, which means that

$$\begin{aligned}
g_x^o &= \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi r(r+Q_x^o)} - 2\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 (h_{y\zeta}^o)^2) \\
2g_{xx}^o &= \frac{A_x}{r+Q_x} + \gamma_M (\sigma_D^2 + \sigma_\zeta^2 (h_{y\zeta}^o)^2) \\
g_{x\zeta}^o &= \frac{1}{r+\kappa} \left( A_\zeta + \frac{A_x Q_\zeta^o}{r+Q_x^o} \right)
\end{aligned}$$

From here, using the solution in 2, we can solve for the trading strategy (i.e.  $\mathbf{Q}$ ). We have the following Proposition:

**Proposition 2.** *Suppose that*

$$\frac{1}{2r\phi} > (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)$$

*then the Markov Perfect Equilibrium with observable shocks is given by*

$$Q_0^o = \frac{r^2 \gamma_M (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)}{(2\phi)^{-1} - r (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)} \quad (8a)$$

$$Q_x^o = \frac{r^2 (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)}{(2\phi)^{-1} - r (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)} \quad (8b)$$

$$Q_\zeta^o = (r + Q_x^o) (2g_{x\zeta}^o (r + \kappa) \phi - \psi). \quad (8c)$$

Using the solutions in Lemma 2, it can be verified that  $Q_\zeta^o < 0$ . This means that a positive shock to the blockholder's ability induces the blockholder to sell shares. To understand this result, notice that  $\zeta_t$  could be interpreted as an endowment shock that increases the exposure of the blockholder to the firm's dividends  $\mu_D + a_t$ . Of course, it's not merely an endowment shock since it also has an impact on the firm's cash flows.

Risk aversion, under CARA preferences, induces the blockholder to sell shares in the face of a positive ability shock. This means that the potential productivity benefits associated with the blockholder holding a larger stake when his ability goes up do not fully materialize, and have a rather transitory effect. The lack of commitment on the part of the blockholder explains this result.

To further understand this issue consider the long-run stock holdings. The steady state mean blockholder holding is

$$\bar{X}_{ss}^o = \frac{\gamma_M}{\gamma_L + \gamma_M}, \quad (9)$$

which coincides with that in DeMarzo and Urošević (2006). Notice that the mean holding depends only on relative risk aversions, but it's independent of the intensity of the moral hazard problem, as measured by  $\phi$ , which suggests that this solution may entail very inefficient levels of effort.

As a special case, we can immediately recover the equilibrium when ability is constant, which corresponds to the solution in DeMarzo and Urošević (2006). Setting  $\psi = 0$ , and picking the solution with  $g_{\zeta\zeta} = g_{x\zeta} = Q_\zeta = 0$  ( $\zeta_t$  is not payoff relevant when  $\psi = 0$  so  $Q_\zeta = 0$  in any Markov



Perfect Equilibrium) we obtain

$$Q_0 = \frac{r^2 \gamma_M \sigma_D^2}{(2\phi)^{-1} - r(\gamma_L + \gamma_M) \sigma_D^2}$$

$$Q_x = \frac{r^2 (\gamma_L + \gamma_M) \sigma_D^2}{(2\phi)^{-1} - r(\gamma_L + \gamma_M) \sigma_D^2},$$

an the price coefficient

$$P_x = \frac{\phi^{-1} - 2r\gamma_L\sigma_D^2}{r}.$$

If the blockholder could commit to a trading strategy at the outset, the optimal solution would resemble that of a contract like [Holmstrom and Milgrom \(1987\)](#). In fact, when  $\gamma_M = 0$  the solution with commitment is exactly that of [Holmstrom and Milgrom \(1987\)](#): the large investor holds a constant fraction of the firm shares and exerts effort  $1/[\phi(1 + \phi\gamma_L\sigma_D^2)]$ , compared to zero effort in the case without commitment (see [DeMarzo and Urošević \(2006\)](#)). But the commitment policy is not time consistent. If the blockholder initially owns all the shares, his marginal valuation would be lower than that of the market, but he would refrain from selling all the way to the price taking solution, because of the impact this strategy would have on the price of his shares. Instead, he would sell gradually. However, once the blockholder sells a share, he no longer internalizes the effect of selling an extra share on the value of the share owned by the first-round buyer. The blockholder sells gradually till he reaches the price taking solution, at which point he no longer has an incentive to exert effort. This inefficiently low effort, caused by lack of commitment, is reminiscent of the Coase conjecture.

Finally, we briefly discuss what happens if the condition  $Q_x^o > 0$  is violated. In [DeMarzo and Urošević \(2006\)](#), when this condition is violated, the large shareholder jumps immediately to the competitive solution, with  $X_t = \bar{X}_{ss}^o$ . In our case, the competitive solution is not constant due to the shocks to  $\zeta_t$ . To illustrate this point, let's consider the maximal solution in [Lemma 2](#). From here, we get that  $Q_x^o$  and  $Q_\zeta^o$  diverge to infinity when the denominator of  $Q_x^o$  becomes zero. What happens to the solution of  $X_t$ ? The solution for  $X_t$ , given the coefficients  $Q_x^o$  and  $Q_\zeta^o$  is given in [Equation \(18\)](#). If we take the limit such that  $Q_x^o$  and  $Q_\zeta^o$  go to infinity and  $Q_\zeta^o/Q_x^o$  converges to a constant  $\tilde{Q}_\zeta^o$ , then we get that

$$X_t = \bar{X}_{ss}^o + \tilde{Q}_\zeta^o \zeta_t.$$

In our case, using Lemma 2 and Equation (8c), we get that  $\tilde{Q}_\zeta^o$  is

$$\begin{aligned}\tilde{Q}_\zeta^o &= 2g_{x\zeta}^o (r + \kappa) \phi - \psi \\ &= -\psi \frac{\sqrt{(r + 2\kappa)^2 + \frac{2r\sigma_\zeta^2\psi^2\gamma_L}{\phi}} - (r + 2\kappa)}{r + \sqrt{(r + 2\kappa)^2 + \frac{2r\sigma_\zeta^2\psi^2\gamma_L}{\phi}}}.\end{aligned}$$

The behavior of the blockholder holding in the observable and unobservable case are fundamentally different. If shocks are observable, the blockholder adjusts his holdings instantly in response to a shock. This is a consequence of the Coasian dynamics highlighted by DeMarzo and Urošević (2006). With asymmetric information, that is no longer the case. As we show in the next section, the incentive to signal high or low productivity leads the blockholder to refrain from trading fast and generates an equilibrium with smooth trading.

*Remark 1.* Two aspects of the previous solution are worth noting. First, notice that the mean stationary holdings when  $\psi = 0$  is the same as the one when  $\psi > 0$  and  $\zeta_t$  is observable. Hence, time varying ability only affects average long term holdings when there is information asymmetry. Also, even though in our continuous time formulation the price impact,  $P_x$ , and long term holding is the same as the one in DeMarzo and Urošević (2006), the rate of trade is higher. In fact, the rate of trade in (DeMarzo and Urošević, 2006, Equation 24 in p. 797) is

$$Q_x = \frac{r^2 (\gamma_L + \gamma_M) \sigma_D^2}{(2\phi)^{-1} - r\gamma_L\sigma_D^2}.$$

Both expressions coincide only if the market is risk neutral ( $\gamma_M = 0$ ). The general lack of convergence between the discrete time limit and the continuous time solution arise here because in continuous time the order flow does not increase the risk exposure of the blockholder (unlike the stock) because he can unwind the order flow quickly enough. Consistent with this, the rate of trade is higher than in the discrete time limit.

## 5 Equilibrium with Asymmetric Information

We return to the general case in which the blockholder's ability  $\zeta_t$  is unobservable. This case poses some challenges. To be able to value the firm shares, the market must infer the evolution of  $\zeta_t$  based on the blockholder's order flow  $q_t^L$ , because the firm's productivity is linked to  $\zeta_t$ . In turn, this may create incentives for the blockholder to manipulate the market beliefs via trading.

Consider the blockholder's problem. Since  $X_t$  and  $q_t$  are observable, the market forms its belief

$\hat{\zeta}_t$  by inverting the blockholder's trading strategy as follows:

$$\hat{\zeta}(q_t, X_t) = \frac{q_t - Q_0 + Q_x X_t}{Q_\zeta} \quad (10)$$

Substituting  $\hat{\zeta}(q_t, X_t)$  in the price function yields

$$p(X_t, \hat{\zeta}_t) = P_0 + P_x X + P_\zeta \hat{\zeta}(q_t, X_t). \quad (11)$$

Hence, the residual supply function faced by the blockholder can be written as

$$R(q_t, X_t) = R_0 + R_x X_t + R_q q_t. \quad (12)$$

This function captures the price facing the blockholder as a function of his order flow.

Unlike the case with observable ability, the price the blockholder pays does not depend on  $\zeta_t$  directly, but only indirectly via the order flow. In general, the more relevant the blockholder ability, as measured by  $\psi$ , the more sensitive is the price to the order flow  $q_t^L$ . This means that the liquidity faced by the blockholder decreases when  $\zeta_t$  is unobservable, particularly so when his ability is more relevant to the firm.

The vector of coefficients  $\mathbf{R}$  is linked to the price function as follows:

$$\begin{aligned} R_0 &= P_0 - \frac{P_\zeta}{Q_\zeta} Q_0, \\ R_x &= P_x + \frac{P_\zeta}{Q_\zeta} Q_x, \\ R_q &= \frac{P_\zeta}{Q_\zeta}. \end{aligned}$$

We can now present the blockholder's problem as:

$$\begin{aligned} &\max_{c, q^L, a} \mathbb{E}_t^L \left[ \int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right] \\ &\text{subject to} \\ &dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - R(X_t, q_t^L)q_t^L + (\mu_D + a_t)X_t)dt + X_t \sigma_D dB_t^D \\ &dX_t = q_t^L dt \end{aligned}$$

The blockholder faces a similar problem as in the observable case except that, in choosing his trading strategy, he must take into account the signaling effect of his order flow; namely, its price

implications. As with the market makers, we conjecture that the value function of the blockholder takes the form

$$V(W, X, \zeta) = -\exp(-r\gamma_L(W + G(X, \zeta))) / r,$$

where the blockholder's certainty equivalent  $G$  satisfies the following HJB equation:

$$rG = \max_{q,a} (\mu_D + a)X - \phi a^2 + \psi \zeta a - R(X, q)q - \frac{1}{2}r\gamma_L (\sigma_D^2 X^2 + \sigma_\zeta^2 G_\zeta^2) + qG_X - \kappa \zeta G_\zeta + \frac{1}{2}\sigma_\zeta^2 G_{\zeta\zeta} \quad (13)$$

Taking the first order conditions, yield the effort and trading strategy of the block holder:

$$a = \frac{\psi \zeta + X}{2\phi}$$

$$q = \frac{G_X - R_0 - R_x X}{2R_q}.$$

For market makers, the stock price is always equal to their marginal valuation of the stock price ( $P = H_Y$ ). In contrast, there is a gap between the blockholder's marginal valuation  $G_X$  and the stock price  $R(q, X)$ . This gap is given by the price effect of the blockholder's order flow. Indeed, we can rewrite the first order condition above as  $G_X - R(X, q) = R_q(X, q)q$ .

In contrast to the observable case—in which the blockholder trades at a competitive price—the presence of private information mitigates the blockholder's commitment problem, and moderates his tendency to trade fast.

The second order condition is satisfied if  $R_q > 0$ , that is, if the residual supply has a positive slope. The less liquid the market faced by the blockholder, the larger the gap between his marginal valuation and the price he faces.

The next result characterizes the blockholder's certainty equivalent as a quadratic function of the two state variables  $\zeta_t$  and  $X_t$ .

**Lemma 3.** *The large shareholder's certainty equivalent is given by*

$$G(\zeta, X) = g_0 + g_x X + g_\zeta \zeta + g_{xx} X^2 + g_{\zeta\zeta} \zeta^2 + g_{x\zeta} X \zeta,$$

where

$$rg_0 = \frac{g_x(g_x - 2R_0)}{4R_q} + \frac{R_0^2}{4R_q} + \sigma_\zeta^2 g_{\zeta\zeta} - \frac{1}{2} r \sigma_\zeta^2 \gamma_L g_\zeta^2 \quad (14a)$$

$$rg_x = \mu_D - r\gamma_L \sigma_\zeta^2 g_\zeta g_{x\zeta} + \frac{(R_0 - g_x)(R_x - 2g_{xx})}{2R_q} \quad (14b)$$

$$(r + \kappa)g_\zeta = \frac{g_{x\zeta}(g_x - R_0)}{2R_q} - 2r\gamma_L \sigma_\zeta^2 g_\zeta g_{\zeta\zeta} \quad (14c)$$

$$(r + \kappa)g_{x\zeta} = \frac{\psi}{2\phi} + \frac{g_{x\zeta}(2g_{xx} - R_x)}{2R_q} - 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta} g_{x\zeta} \quad (14d)$$

$$(r + 2\kappa)g_{\zeta\zeta} = \frac{\psi^2}{4\phi} + \frac{g_{x\zeta}^2}{4R_q} - 2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^2 \quad (14e)$$

$$rg_{xx} = \frac{1}{4\phi} + \frac{(2g_{xx} - R_x)^2}{4R_q} - \frac{1}{2} r\gamma_L (\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2) \quad (14f)$$

We can then use the first order conditions to obtain the coefficients  $\mathbf{Q}$  as given by

$$Q_0 = \frac{g_x - R_0}{2R_q}$$

$$Q_x = \frac{R_x - 2g_{xx}}{2R_q}$$

$$Q_\zeta = \frac{g_{x\zeta}}{2R_q}$$

On the other hand, using the previous coefficients together with the equations for  $R_0$  and  $R_q$ , we can write the coefficients of the price function in terms of  $Q_0$  and  $Q_\zeta$ , which are given by

$$P_0 = g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} \quad (15a)$$

$$P_x = 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \quad (15b)$$

$$P_\zeta = \frac{g_{x\zeta}}{2}. \quad (15c)$$

At the same time, from the solution to the market maker problem the price coefficients must satisfy Equation (3a)-(3c). In equilibrium, both sets of coefficients must be the same, so combining

(3a)-(3c) and (15a)-(15c) we find that

$$g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} = \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r + Q_x)} - \gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \quad (16a)$$

$$2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} = \frac{1}{2\phi(r + Q_x)} + \gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \quad (16b)$$

$$g_{x\zeta} = \frac{1}{r + \kappa} \left( \frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r + Q_x)} \right). \quad (16c)$$

A linear equilibrium is given by the solution to equations (14a)-(14f) along with equation (16a)-(16c).

**Proposition 3.** *There exists a linear Markov perfect equilibrium with smooth trading if and only if the polynomial system*

$$\begin{aligned} (r + \kappa)g_{x\zeta} &= \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta})g_{x\zeta} \\ (r + 2\kappa)g_{\zeta\zeta} &= \frac{Q_\zeta}{2} g_{x\zeta} - 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \\ \frac{Q_x}{2Q_\zeta} (r + 2Q_x) g_{x\zeta} &= -\frac{Q_x}{2\phi(r + Q_x)} + r(\gamma_L + \gamma_M)\sigma_D^2 + r \left( \frac{\gamma_M}{4} + \gamma_L \right) \sigma_\zeta^2 g_{x\zeta}^2 \\ g_{x\zeta} &= \frac{1}{r + \kappa} \left( \frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r + Q_x)} \right) \end{aligned}$$

admits a solution with  $Q_x > 0$  and  $R_q = g_{x\zeta}/2Q_\zeta > 0$ .

Notice that Equation (16c) implies that any solution with positive  $Q_x, Q_\zeta$  immediately satisfies the condition  $R_q > 0$ , so it is an equilibrium. Depending on the importance of the blockholder's ability  $\psi$ , there are up to three equilibria, as discussed in the next section. Existence, requires that the residual supply is downward sloping (i.e.  $R_q > 0$ ). It also requires that the blockholder's inventory converges to a stationary distribution, which only happens if the order-flow decreases in the blockholder's inventory,  $X_t$ .

To obtain the equilibrium, we need to solve a system of four polynomial equations. In general, this system has no closed form. As in Vayanos (1999, 2001) we can solve for the equilibrium when the volatility of shocks is small. We consider the small noise limit when  $\sigma_\zeta^2$  and  $\kappa$  go to zero but  $\sigma_\zeta^2/2\kappa \rightarrow \bar{\sigma}_\zeta^2 > 0$ . This corresponds to a situation when ability shocks are small but highly persistent.

**Proposition 4.** Consider the small noise limit  $\kappa, \sigma_\zeta^2 \rightarrow 0$ ,  $\sigma_\zeta^2/2\kappa \rightarrow \bar{\sigma}_\zeta^2 > 0$ . In the limit, there is an equilibrium with liquidity

$$R_q = \frac{\sqrt{(\alpha + 1)^2 + 8\alpha^2} - \alpha - 1}{4r\phi}$$

where  $\alpha \equiv r\phi(\gamma_L + \gamma_M)\sigma_D^2$ . The coefficients of the trading strategy are

$$Q_x = \frac{rR_q}{(\gamma_L + \gamma_M)\sigma_D^2}$$

$$Q_\zeta = \frac{\psi}{2\phi(\gamma_L + \gamma_M)\sigma_D^2},$$

and the coefficients of the price function are

$$P_x = R_q + \frac{1}{2r\phi} + \gamma_M\sigma_D^2$$

$$P_\zeta = \frac{\psi}{4r\phi} \frac{(\gamma_L + \gamma_M)\sigma_D^2}{R_q}$$

The steady-state mean stock-holding of the blockholder is

$$\bar{X}_{ss} = \left( 1 + \frac{\gamma_L\psi^2\bar{\sigma}_\zeta^2}{\phi + \gamma_L\psi^2\bar{\sigma}_\zeta^2} \frac{rR_q}{\phi(\gamma_L + \gamma_M)^2\sigma_D^4} \right) \frac{\gamma_M}{\gamma_L + \gamma_M}.$$

This limit case admits a closed-form solution. An equilibrium with signaling always exists. It features a positive trading sensitivity to ability ( $Q_\zeta > 0$ ), and a positive price sensitivity to ability ( $P_\zeta > 0$ ). By contrast: if ability  $\zeta$  were observable, the blockholder's trading would not respond to ability shocks, namely  $Q_\zeta = 0$ .

The market's liquidity  $R_q^{-1}$  decreases in risk aversion  $\gamma$  and the volatility of cash flows  $\sigma_D^2$  but it is independent of  $\psi$ . Liquidity also decreases in the severity of moral hazard  $\phi$ . Proposition 4 shows that the blockholder's average holdings are larger than in the observable case; more so the larger the volatility of ability ( $\sigma_\zeta^2$ ). The large long-run blockholder's holdings is driven by signaling incentives under risk aversion ( $\gamma_M > 0$ ). In other words, price impact plays a key role behind. Moral hazard is not essential; this large holding result even as we remove hidden effort from the model.

## 5.1 Multiplicity and Comparative Statics

The feedback between stock prices and firm productivity may generate multiplicity, particularly when the importance of the blockholder ability,  $\psi$ , is low. This result is reminiscent of the feedback effects analyzed by [Edmans et al. \(2015\)](#). There is an equilibrium with low liquidity, where the

blockholder refrains from aggressively trading based on his private information and the price is also relatively insensitive to the ability shocks. On the other extreme, there is an equilibrium with high liquidity where the blockholder trades aggressively based on his private information and the price is relatively sensitive to ability shocks.

The multiplicity of equilibria depends on  $\psi$ . For low  $\psi$ , there are three equilibria, two of which entail a negative  $P_\zeta$ . For large  $\psi$ , the equilibrium is unique. In this section, we discuss the properties of these equilibria. In general, computing the set of all equilibria is involved. In a linear quadratic model, as the one we analyze, the set of equilibrium conditions reduces to a system of polynomial equations that can be analyzed using tools from computational algebraic geometry (Kubler et al. (2014) provides an introduction to these techniques and to its use to study multiple equilibria in economics).<sup>8</sup> The equilibrium correspondence, as a function of  $\psi$ , is depicted in Figure 1.

If we look at the the plot of  $Q_\zeta$  in Figure 1, we see that for low  $\psi$  there are three equilibria, and two of them feature  $Q_\zeta < 0$ , consistent with the observable case. One of these bottom equilibria yields  $Q_\zeta = 0$  as  $\psi \downarrow 0$ , thus converging to the unique Markov equilibrium arising when  $\zeta_t$  is observable. By contrast, the very bottom equilibrium, converges to  $Q_\zeta < 0$ , which represents a situation where trading depends on the blockholder’s ability even though it is payoff irrelevant; hence in the limit, this is not a Markov equilibrium. The upper branch of the correspondence, depicts an equilibrium with  $Q_\zeta > 0$ . This is the only equilibrium that survives when  $\psi$  is large, which is the case we are interested about, and the one we focus henceforth. By looking at the plot of  $R_q$  in Figure 1, we see that the equilibrium with  $Q_\zeta > 0$  features the highest liquidity (i.e., lowest  $R_q$ ).

## 5.2 Dynamics and Steady State

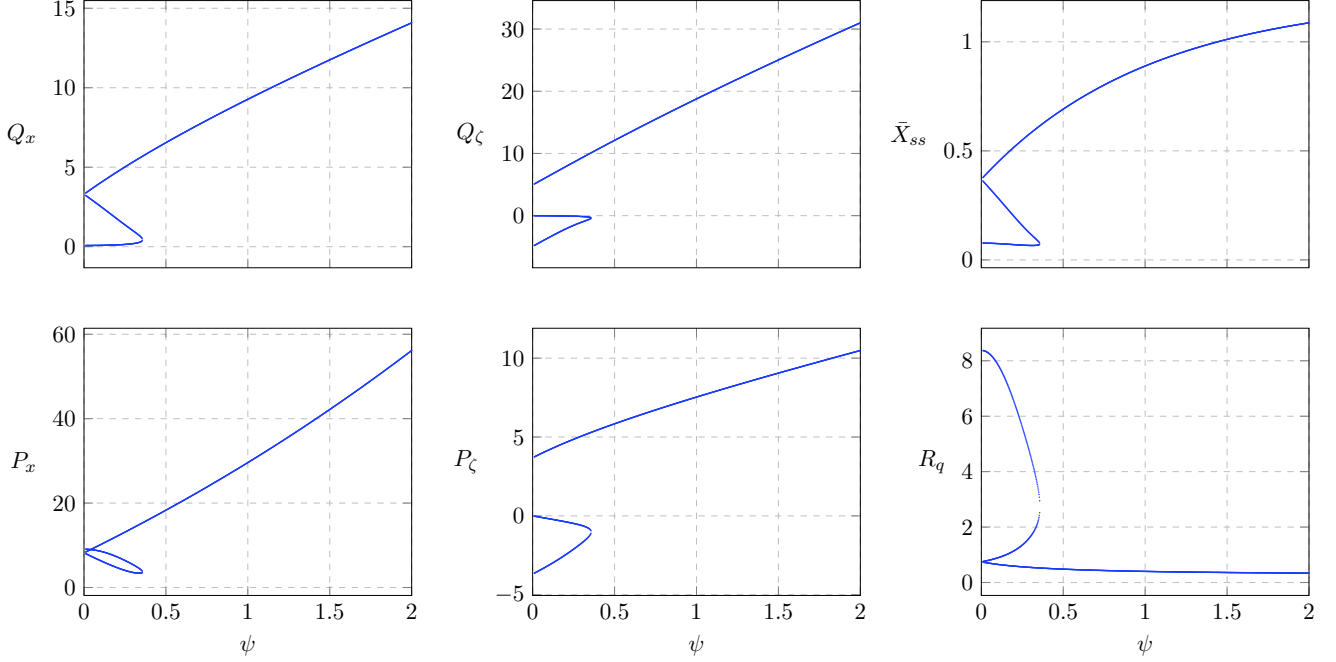
Our model provides a laboratory to examine the effect of activism on the dynamics of firm value and asset prices. In this section, we study such dynamics. First, we analyze the steady state distribution of the two state variables,  $X_t$  and  $\zeta_t$ . Then we study the impulse response functions of excess returns, risk premia, and effort.

The blockholder’s stock-holdings is determined by the solution to the following linear system of stochastic differential equations:

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<sup>8</sup>Specifically, we use Gröbner bases to find all the solutions to the polynomial system in Proposition 3. Gröbner bases is a generalization of Gaussian elimination to polynomial systems. It allows to reduce the problem of finding the solution of a multivariate polynomial to finding the solution of a univariate polynomial. Moreover, the reduction to the univariate polynomial is exact so the only step in the analysis that requires numerical approximation is the solution to the univariate polynomial. A problem for which there are efficient and reliable methods.





**Figure 1:** Set of Equilibria for different values of  $\psi$ . Parameter values:  $\gamma_M = 0.5$ ,  $\gamma_L = 6$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\kappa = 0.35$ ,  $\phi = 0.5$ ,  $r = 0.05$ ,  $\mu_D = 1$ .

$$\begin{pmatrix} dX_t \\ d\zeta_t \end{pmatrix} = \left[ \begin{pmatrix} Q_0 \\ 0 \end{pmatrix} + \begin{pmatrix} -Q_x & Q_\zeta \\ 0 & -\kappa \end{pmatrix} \begin{pmatrix} X_t \\ \zeta_t \end{pmatrix} \right] dt + \begin{pmatrix} 0 \\ \sigma_\zeta \end{pmatrix} dB_t^\zeta. \quad (17)$$

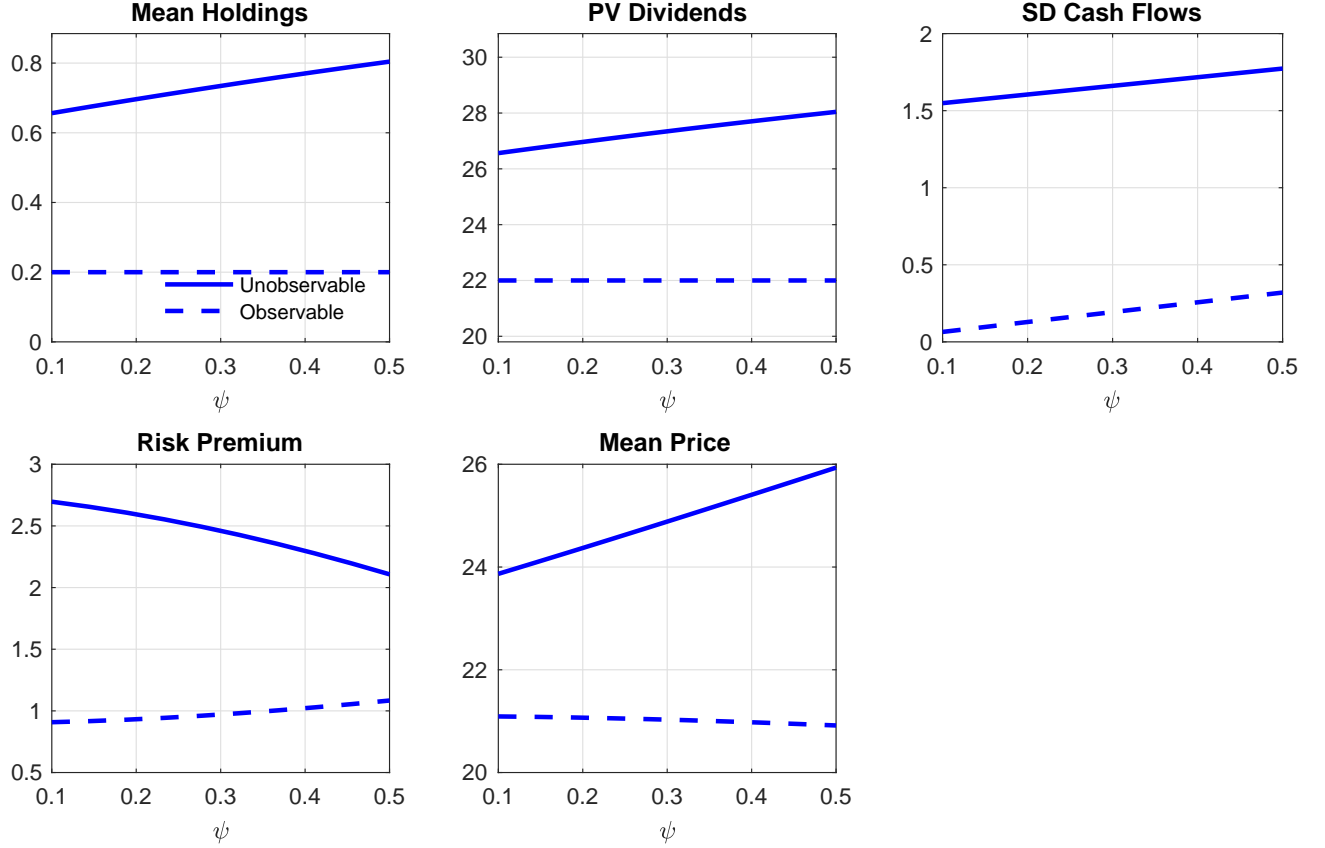
The solution for  $X_t$  is the following (see, e.g. [Evans \(2012\)](#)):

$$X_t = \bar{X}_{ss} + e^{-Q_x t} (X_0 - \bar{X}_{ss}) + \frac{(e^{-\kappa t} - e^{-Q_x t}) Q_\zeta \zeta_0}{Q_x - \kappa} + \int_0^t \frac{Q_\zeta (e^{-\kappa(t-s)} - e^{-Q_x(t-s)})}{Q_x - \kappa} \sigma_\zeta dB_s^\zeta, \quad (18)$$

where  $\bar{X}_{ss} = Q_0/Q_x$  is the long-run mean of  $X_t$ . Based on this equation, we can compute the stationary distribution of  $(X_t, \zeta_t)$ , which is given by

$$(X_t, \zeta_t) \xrightarrow[t \rightarrow \infty]{\mathcal{D}} N \left[ \begin{pmatrix} \bar{X}_{ss} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sigma_\zeta^2 Q_\zeta^2}{2\kappa Q_x (Q_x + \kappa)} & \frac{\sigma_\zeta^2 Q_\zeta}{2\kappa (Q_x + \kappa)} \\ \frac{\sigma_\zeta^2 Q_\zeta}{2\kappa (Q_x + \kappa)} & \frac{\sigma_\zeta^2}{2\kappa} \end{pmatrix} \right]. \quad (19)$$

Figure 2 plots the economic consequences of  $\psi$  for both the observable and unobservable ability cases. In the unobservable case, the blockholder ability  $\psi$  has a relatively strong effect: as the importance of the blockholder's ability goes up, his mean stock-holdings increase. By contrast, in the observable case, the blockholder's mean stock-holdings are independent of  $\psi$ ; instead they are



**Figure 2:** Steady State. Parameters:  $\gamma_M = 0.5$ ,  $\gamma_L = 2$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\kappa = 0.3$ ,  $\phi = 1$ ,  $r = 0.05$ ,  $\mu_D = 1$ .

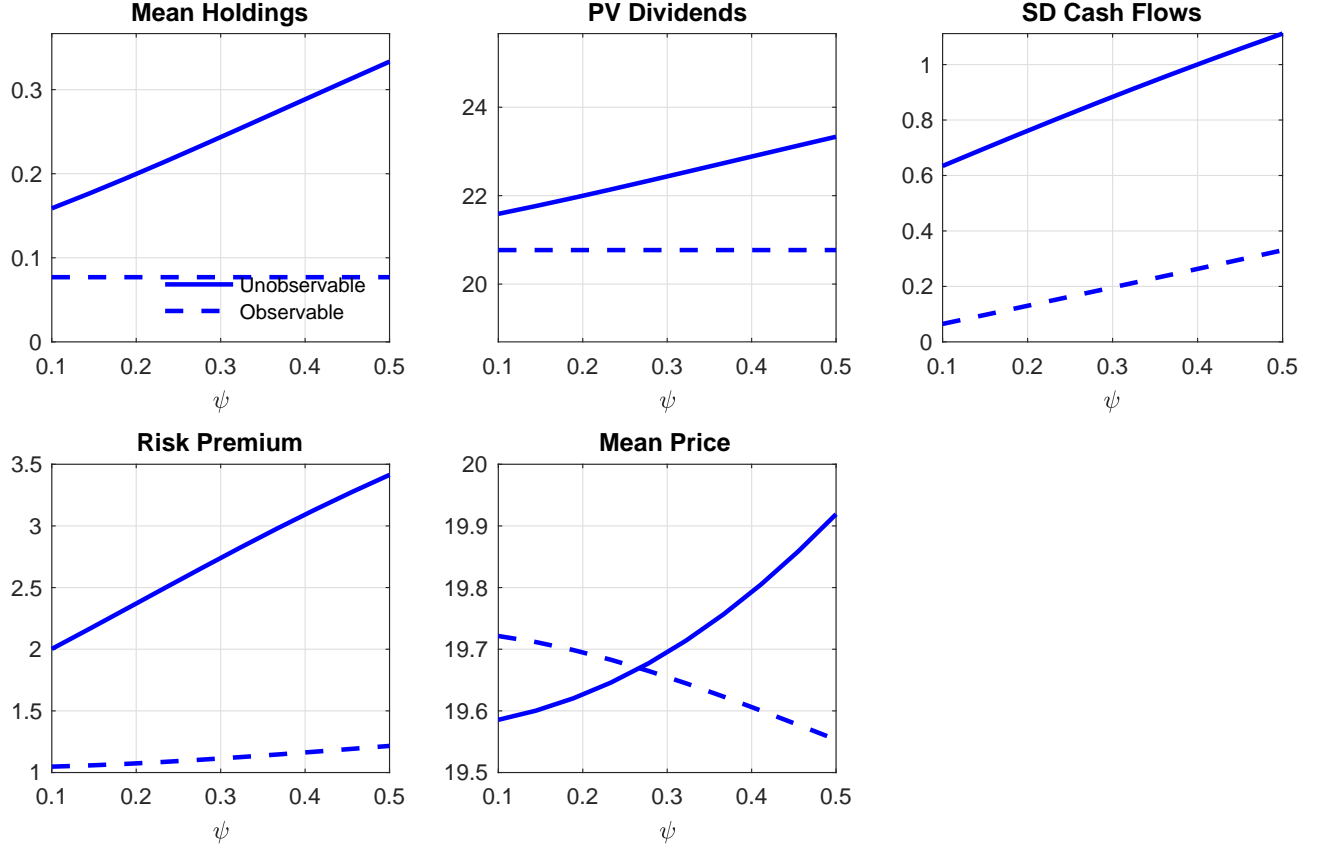
determined by the market's risk-bearing capacity.

In the unobservable case, the firm tends to be more productive; the present value of dividends goes up as activism intensifies. However, effort and dividends are also more volatile, which explains why the risk-premium is higher. The overall price impact of unobservable  $\zeta$  is ambiguous and depends on the blockholder's risk aversion. When his risk aversion is low, the price is higher in the unobservable case, so the productivity effect dominates. However, when the blockholder's risk aversion is relatively high, the price may be lower relative to the observable case, so the risk effect dominates (see Figure 3).

The next proposition provides some comparative statics for stationary distribution in the small noise limit.

**Proposition 5.** *In the small noise limit equilibrium in Proposition 4:*

1. The mean steady-state block  $\bar{X}_{ss}$  is increasing in  $\psi$  and  $\bar{\sigma}_\zeta^2$  and decreasing in  $\phi$  and  $\sigma_D^2$ .



**Figure 3:** Steady State. Parameters:  $\gamma_M = 0.5$ ,  $\gamma_L = 6$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\kappa = 0.3$ ,  $\phi = 1$ ,  $r = 0.05$ ,  $\mu_D = 1$ .

2. The mean steady-state price,  $\bar{p}_{ss} = P_0 + P_x \bar{X}_{ss}$ , is increasing in  $\psi$  and  $\bar{\sigma}_\zeta^2$ , and decreasing in  $\phi$ . If  $\bar{X}_{ss} < 1$  then the mean price is also decreasing in  $\sigma_D^2$ .
3. The stationary variance of  $X_t$  is

$$\mathbb{V}[X_t] = \left( \frac{2\bar{\sigma}_\zeta \psi}{\sqrt{(\alpha + 1)^2 + 8\alpha^2} - \alpha - 1} \right)^2,$$

$\alpha \equiv r\phi(\gamma_L + \gamma_M)\sigma_D^2$ . Hence, the long run variance of  $X_t$  is:

- (a) Increasing in  $\psi$  and  $\bar{\sigma}_\zeta$ .
- (b) Decreasing in  $\phi$ ,  $\gamma_L + \gamma_M$  and  $\sigma_D^2$ .

This result shows that, in the long-run, the blockholder holdings are higher when the blockholder ability is more volatile (i.e. as  $\bar{\sigma}_\zeta^2$  goes up) which in turn boosts the average stock-price. In contrast, when ability is observable, the average stock price would neither depend on  $\sigma_\zeta$  or  $\psi$ .

We can study the dynamics of our problem by looking at the impulse response functions. Specifically, we look at the evolution of equilibrium outcomes after the realization of a one standard deviation shock to the blockholder's ability.

The impulse response function of  $X_t$  is given by the coefficient in the stochastic integral in Equation (18). This equation is the continuous time analogous to the standard moving average representation in discrete time VAR models. In fact, in our model the blockholder stock holdings follows the equivalent of an AR(2) process. The impulse response function for the large shareholder holding is

$$\sigma_\zeta^{-1} IR_X(t) = \frac{Q_\zeta}{Q_x - \kappa} (e^{-\kappa t} - e^{-Q_x t}) \quad (20)$$

$IR_X(t)$  captures the evolution of  $X_t$  over time, after a one time increase in ability  $\zeta_t$ . The Impulse response of ability is

$$\sigma_\zeta^{-1} IR_\zeta(t) = e^{-\kappa t} \quad (21)$$

Based on these impulse responses, we can characterize the evolution of all the variables of interest. For example, we can derive the impulse response of effort, as given by

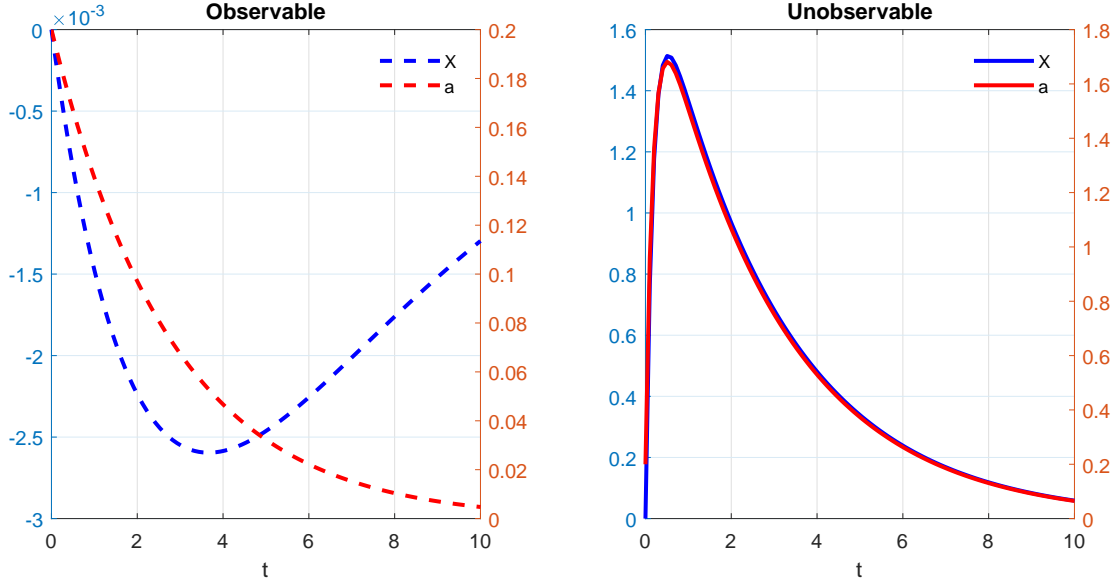
$$\sigma_\zeta^{-1} IR_a(t) = \left( \frac{Q_\zeta}{2\phi(Q_x - \kappa)} + \frac{\psi}{2\phi} \right) e^{-\kappa t} - \frac{Q_\zeta}{2\phi(Q_x - \kappa)} e^{-Q_x t} \quad (22)$$

Figure 4 plots these functions. The right panel depicts the unobservable case. Initially, in the face of a positive shock the blockholder buys more shares. He holds these new shares for some time, and then unloads them as the shock dissipates, given the ability's mean reversion. Effort is affected both directly via the change in ability  $\zeta_t$ , and indirectly via the increase in the blockholder stock-holdings. The more shares the blockholder buys the stronger his incentive to exert effort, as he internalizes the benefits of effort more fully.

The left panel depicts the observable case. When the shock to ability  $\zeta_t$  is observable, the stock holdings remain virtually unchanged in response to the shock realization. Also, relative to the unobservable case, the dynamic pattern is qualitatively different: Initially, the blockholder sells shares. This is driven by risk aversion: a shock to the blockholder's ability is similar to an endowment shock. Observe, indeed, that the stock's dividend is  $\mu_D + a_t$  and the blockholder cost is  $\phi a_t^2 - \zeta a_t$ . Given the blockholder's CARA preferences, a positive shock to ability, increases the stock price in anticipation of higher dividends, and induces the blockholder to sell some shares—despite his greater productivity—to reduce his risk exposure. Thus, the blockholder reduces his stock holdings precisely when his effort becomes more desirable from the perspective of the firm's shareholders.

On the other hand, the dynamics of effort, in the observable case, are dominated by the di-

rect effect of  $\zeta_t$  given that the blockholder's holding  $X_t$  remain virtually unchanged: effort jumps upwards and its level goes down monotonically until the ability shock goes away.



**Figure 4:** Impulse Response: Ownership and Effort. Parameters:  $\gamma_M = 0.5$ ,  $\gamma_L = 6$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\kappa = 0.35$ ,  $\phi = 0.5$ ,  $\psi = 0.2$ ,  $r = 0.05$ ,  $\mu_D = 1$ .

Next, we discuss the dynamics of asset pricing. In particular, we focus on the dynamics of the risk premium. When investors have CARA utility the price equals the present value of dividends, discounted at the risk free rate, net of the risk premium. As [Campbell and Kyle \(1993\)](#) put it, under CARA, “investors demand a risk premium per share of stock rather than per dollar invested.” Consistent with this, we calculate the risk premium indirectly from the present value of dividends  $PV_t$  net of the stock price  $P_t$ . In other words, we define the risk premium as  $RP_t \equiv PV_t - P_t$ . The present value of dividends discounted at the risk free rate. Let

$$\begin{aligned} PV_t &= \mathbb{E}_t^M \left[ \int_0^\infty e^{-r(s-t)} dD_s \right] \\ &= \mathbb{E}_t \left[ \int_0^\infty e^{-r(s-t)} (\mu_D + A_\zeta \zeta_s + A_x X_s) ds. \right] \end{aligned} \quad (23)$$

We can calculate  $PV$  using the solution for  $X_t$  in Equation (18). This yields

$$PV_t = \frac{\mu_D}{r} + \frac{A_x Q_x}{r(r + Q_x)} \bar{X}_{ss} + \frac{A_x}{r + Q_x} X_t + \frac{A_\zeta (r + Q_x + Q_\zeta)}{(r + \kappa)(r + Q_x)} \zeta_t. \quad (24)$$

Naturally, the present value of dividends is sensitive to  $\zeta_t$  since a higher ability stimulates the

blockholder's effort, as he becomes more productive (effort is less costly). The dividends are also sensitive to  $X_t$  because the blockholder exerts more effort when he owns more shares (the free riding problem is weaker).

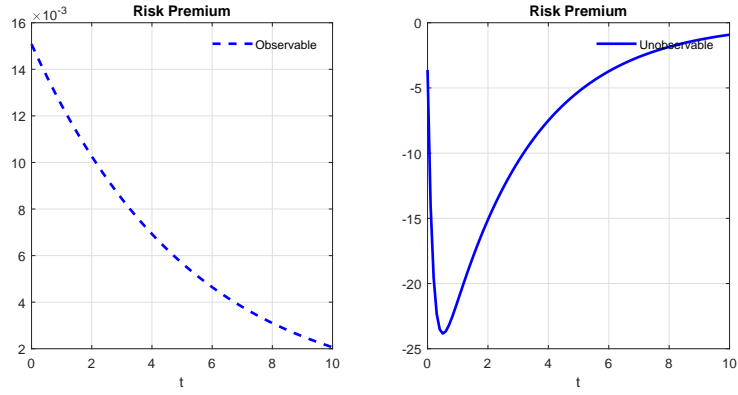
Figure 5, examines the impulse response of the dividends  $PV_t$ , with and without information asymmetry about  $\zeta$ . In both cases, the shock generates a positive effect on the dividends, but the effect is stronger in the unobservable case, given its relatively larger impact on the blockholder's stock-holdings.

On the other hand, we have from Proposition 1 that the risk premium is given by

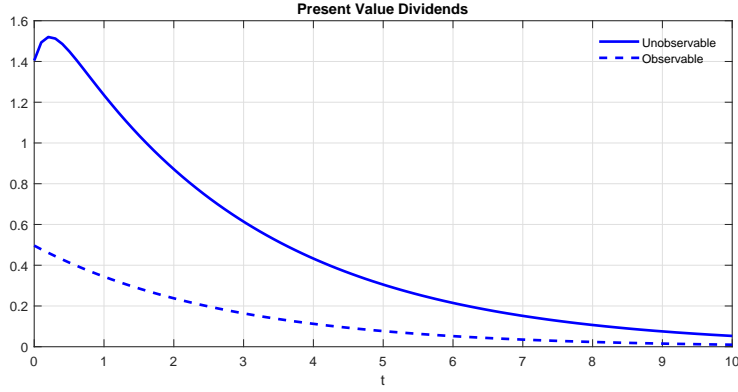
$$\begin{aligned} RP_t &= \mathbb{E}_t^M \left[ \int_t^\infty e^{-r(s-t)} r \gamma_M \hat{\sigma}_D^2 (1 - X_s) ds \right] \\ &= \gamma_M \hat{\sigma}_D^2 \left[ 1 - \frac{1}{2\phi} \left( \frac{Q_x}{r + Q_x} \bar{X}_{ss} + \frac{r}{r + Q_x} X_t \right) \right], \end{aligned} \quad (25)$$

where  $\hat{\sigma}_D^2$  is the variance of long term dividends adjusted by the uncertainty of  $\zeta$  and is defined in Proposition 1. The risk premium is a function of both the ability of the blockholder  $\zeta_t$  and his stock-holdings  $X_t$ . The former, affects the risk exposure of the market (because the price increases in  $\zeta_t$ ) and the latter affects the proportion of the total risk that the market has to absorb. Figure 5, shows the response of the risk premium to the shock, with and without information asymmetry. In the observable case, the risk premium increases, but barely. In the unobservable case, the risk premium experiences a large drop after the shock, reflecting the blockholder's choice to increase his inventory. The risk premium continues to drop for some time while the blockholder gradually increases his stock holdings till the desired level. As the ability shock goes away, the blockholder gradually sells his excess inventory back driving the risk premium up to its steady state level.

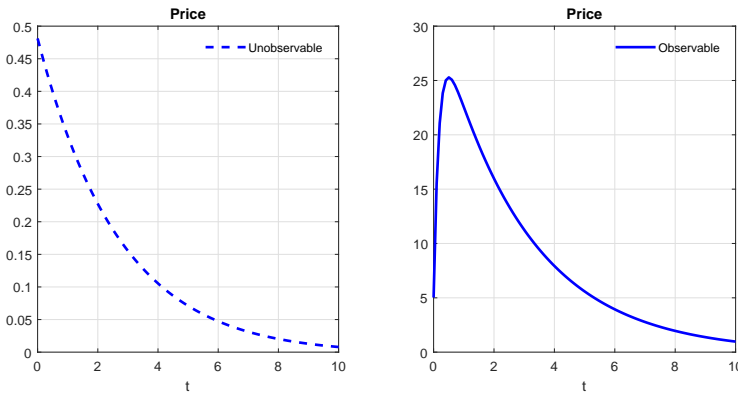
Figure 5 also depicts the evolution of the stock price. We see that in the unobservable case, the shock causes an initial increase in the stock price, driven by higher dividends and lower risk premia, that reverses in the long run, as the shock reverses and the blockholder starts selling back his excess shares toward the steady state. Quantitatively, the most relevant effect driving the stock price dynamics in the unobservable case appears to be the risk premium, compared to the dividends. It is also worth noting, that in the observable case the price barely reacts to the shock, because of the insensitivity of the blockholder's holdings to his ability.



(a) Impulse Response Risk premium



(b) Impulse Response Present Value Dividends



(c) Impulse Response Stock Price

**Figure 5:** Impulse Response: Parameters:  $\gamma_M = 0.5$ ,  $\gamma_L = 6$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\kappa = 0.35$ ,  $\phi = 0.5$ ,  $\psi = 0.2$ ,  $r = 0.05$ ,  $\mu_D = 1$ .

## 6 Private Benefits: Beyond a Fully Revealing Equilibrium

Our baseline model features an equilibrium where the blockholder's trading history fully reveals his ability,  $\zeta_t$ . This means that, on equilibrium, there is no asymmetry of information between the market and the blockholder. Furthermore, the firm cash flow does not provide any incremental information, beyond that provided by blockholder's order flow, which means that the effort choice only depends on the blockholder's position and his ability but not on his reputation.

In this section, we extend the baseline and consider a situation in which the market is not able to perfectly infer the blockholder's productivity from the blockholder's trading. As a result, cash flows are informative about the blockholder's productivity and his incentives are affected by reputation concerns.

In particular, we consider a second source of uncertainty about the blockholder's motive for trade. We assume that the blockholder obtains a private benefit  $b_t$  from owning the firm's shares—in addition to the dividend  $dD_t$ . One possible interpretation is that the blockholder is subject to unobservable liquidity shocks that affect his cost of holding a position in the firm, and that this liquidity shocks are orthogonal to the firm's fundamentals.

Thus, similar to [Manzano and Vives \(2011\)](#), [Hatchondo, Krusell, and Schneider \(2014\)](#) and [Dávila and Parlato \(2017\)](#), we consider a setting in which trading is noisy due to the presence of unobservable endowment shocks rather than due to the presence of noise traders. We depart from traditional models with noise traders because, in practice, the size of the block of an activist or a large shareholder is largely observable (albeit with some delay). Private benefits affect future trading needs, and due to moral hazard, it also affect stock prices and the risk premium in a way that is qualitatively different from the effect of noise traders.

The private benefit is privately observed by blockholder and follows an Ornstein-Uhlenbeck process as given by

$$db_t = -\lambda b_t dt + \sigma_b dB_t^b,$$

where  $\lambda$  captures the persistence of private benefits. In turn, the blockholder's wealth process is given by

$$dW_t = (rW_t - c_t - R_t(q_t)q_t - \Phi(a_t, \zeta_t) + (\mu_D + a_t + \delta b_t)X_t)dt + X_t \sigma_D dB_t^D.$$

The parameter  $\delta$  captures the exposure of the blockholder to the private benefit shock. The baseline model without private benefits corresponds to the case in which  $\delta = 0$ .

Denoting the market beliefs by  $\hat{b}_t = \mathbb{E}[b_t | (D_s, q_s^L)_{s \leq t}]$   $\hat{\zeta}_t = \mathbb{E}[\zeta_t | (D_s, q_s^L)_{s \leq t}]$ , then a linear Markov equilibrium is given by an affine function of the payoff relevant variables  $(X_t, \zeta_t, b_t, \hat{\zeta}_t, \hat{b}_t)$ .



In particular, we consider a stationary linear equilibrium:

$$q_t^L = Q_0 - Q_x X_t + Q_\zeta \zeta_t + Q_b b_t + Q_{\hat{b}} \hat{b}_t \quad (26a)$$

$$a_t = A_0 + A_x X_t + A_\zeta \zeta_t + A_b b_t + A_{\hat{b}} \hat{b}_t \quad (26b)$$

$$p_t = P_0 + P_x X_t + P_\zeta \zeta_t + P_{\hat{b}} \hat{b}_t \quad (26c)$$

$$R(q_t) = R_0 + R_x X_t + R_{\hat{b}} \hat{b}_t + R_q q_t \quad (26d)$$

Notice that our conjectured equilibrium strategies  $q_t^L$  and  $a_t$  are a function of  $\hat{b}_t$  but not  $\hat{\zeta}_t$ . This happens because the expectation  $\hat{\zeta}_t$  is uniquely determined by  $q_t^L$  and  $\hat{b}_t$ .

To find the equilibrium we take the following steps. First, we derive the market's beliefs given the conjectured equilibrium by solving for the market's filtering problem. Next, we solve the competitive investors portfolio problem and derive the residual supply faced by the blockholder. Finally, we solve the blockholder's optimization problem.

Because the market perfectly observes the order flow  $q_t^L$ , its information can be summarized by the variable

$$Z_t \equiv \frac{q_t - Q_0 + Q_x X_t - Q_{\hat{b}} \hat{b}_t}{Q_\zeta} = \zeta_t + \frac{Q_b}{Q_\zeta} b_t, \quad (27)$$

where  $Z_t$  captures the information provided by the blockholder's order flow. From the market perspective, the order flow is a noisy signal about  $\zeta$  because it is also affected by the magnitude of private benefits. The market can not perfectly disentangle what is driving the blockholder trading, namely ability or private benefits.

The filtering problem of the market is non-standard. Unlike in standard Kalman filtering problems, the market observes a linear combination of  $\zeta_t$  and  $b_t$  without any noise, so we cannot use standard filtering techniques. In fact, conditional on  $Z_t$ , the support of beliefs is the line  $Z_t = \zeta_t + (Q_b/Q_\zeta)b_t$  instead of the full support  $\mathbb{R}^2$ , which means that the covariance matrix is singular. Technically, this corresponds to a singular filtering problem (Xiong, 2008).<sup>9</sup> The trick to solve the investors' filtering problem is to transform the original two-dimensional filtering problem for  $(\zeta_t, b_t)$  into a single dimensional filtering problem for  $b_t$ . Then, once we have determined the belief  $\hat{b}_t$ , we can solve for  $\hat{\zeta}_t$  using equation (27) which implies

$$\hat{\zeta}_t = Z_t - \frac{Q_b}{Q_\zeta} \hat{b}_t.$$

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<sup>9</sup>More generally, this is a filtering problem with Ornstein-Uhlenbeck noise. The theory of filtering for general Gaussian process is developed in Kunita (1993). The specific case with Ornstein-Uhlenbeck noise is developed in detail in Liu and Xiong (2010).

In some sense, this is similar to the way we solve for  $\hat{\zeta}_t$  when the order flow is fully revealing, with the exception except that the intercept of the linear function is time varying and determined by  $\hat{b}_t$ .

If we differentiate  $Z_t$ , and use equation (27) to eliminate  $\zeta_t$ , we get the following stochastic differential equation for  $Z_t$

$$dZ_t = -\kappa Z_t dt + (\kappa - \lambda) \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b. \quad (28)$$

Similarly, replacing the conjectured equilibrium effort, and substituting  $\zeta_t$  by using equation (27), we find that the dividend process follows

$$dD_t = \left( \mu_D + A_0 + A_x X_t + A_\zeta Z_t - A_\zeta \frac{Q_b}{Q_\zeta} b_t + A_b b_t + A_b \hat{b}_t \right) dt + \sigma_D d\tilde{B}_t^D. \quad (29)$$

At this point, we have transformed our original singular filtering problem for  $(\zeta_t, b_t)$  into a standard filtering problem for  $b_t$ , in which the observation process consists of the two signals  $(D_t, Z_t)$ . We can use the Kalman-Bucy formula to get the market's belief updating

$$d\hat{b}_t = -\lambda \hat{b}_t dt + \beta_q \left( \sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \right) + \beta_D \sigma_D d\tilde{B}_t^D,$$

where  $(\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D)$  are Brownian motions under the filtration generated by  $(q_t, D_t)_{t \geq 0}$ . In a stationary linear equilibrium, the covariance matrix of  $(\hat{b}_t, \hat{\zeta}_t)$  is constant. Because in our setting we only need to keep track of  $\hat{b}_t$ , this amounts to looking for the stationary solution of the differential equation for the conditional variance of  $b_t$ , which we denote by  $\sigma_b^2 \equiv \mathbb{V}[b_t | \mathcal{F}_t^{q,D}]$ . In the appendix, we show that the evolution of the vector  $(\hat{\zeta}_t, \hat{b}_t)$  is given by the following proposition.

**Proposition 6.**  $\hat{\zeta}_t$  and  $\hat{b}_t$  satisfy the following stochastic differential equation

$$d\hat{b}_t = -\lambda \hat{b}_t dt + \beta_q \left( \sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \right) + \beta_D \sigma_D d\tilde{B}_t^D \quad (30a)$$

$$d\hat{\zeta}_t = -\kappa \hat{\zeta}_t dt + \left( 1 - \beta_q \frac{Q_b}{Q_\zeta} \right) \left( \sigma_\zeta d\tilde{B}_t^\zeta + \sigma_b \frac{Q_b}{Q_\zeta} d\tilde{B}_t^b \right) - \beta_D \frac{Q_b}{Q_\zeta} \sigma_D d\tilde{B}_t^D, \quad (30b)$$

$$d\tilde{B}_t^\zeta = \sigma_\zeta^{-1} (d\zeta_t + \kappa \hat{\zeta}_t dt)$$

$$d\tilde{B}_t^b = \sigma_b^{-1} (db_t + \lambda \hat{b}_t dt)$$

$$d\tilde{B}_t^D = \sigma_D^{-1} (dD_t - (\mu_D + E_t(a_t)) dt)$$

where

$$\beta_q = \frac{\sigma_b^2 + (\kappa - \lambda)\sigma_b^2 \frac{Q_b}{Q_\zeta}}{\left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2 + \sigma_\zeta^2} \frac{Q_b}{Q_\zeta}$$

$$\beta_D = \frac{\sigma_b^2}{\sigma_D^2} \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right)$$

and

$$0 = -2\lambda\sigma_b^2 + \sigma_b^2 - \left[ \frac{\left(\sigma_b^2 + (\kappa - \lambda)\sigma_b^2\right)^2}{\sigma_\zeta^2 + \left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2} \left(\frac{Q_b}{Q_\zeta}\right)^2 - \left(\frac{\sigma_b^2}{\sigma_D^2}\right)^2 \left(A_b - A_\zeta \frac{Q_b}{Q_\zeta}\right)^2 \right]. \quad (31)$$

The innovation processes  $(\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D)$  are standard Brownian motions with respect to the filtration  $(\mathcal{F}_t^M)_{t \geq 0}$ .

The sensitivity of beliefs to unexpected order-flow or dividend shocks depends on the way trading and effort react to private benefit and ability shocks, and the speed of mean reversion of these variables. For example, if the blockholder's order flow is increasing in both  $b_t$  and  $\zeta_t$  ( $Q_\zeta$  and  $Q_b$  are positive), then market beliefs about private benefit ( $\hat{b}_t$ ) will increase after unexpected shocks to trading. This means that part of the increase in trading is attributed to private benefits. The impact on beliefs about ability will depend on how large this reaction is relative to the expected reaction if the increment in trading were driven by productivity shocks. In the case of unexpected dividends, the reaction of beliefs depends on the magnitude of  $A_b/A_\zeta$  relative to  $Q_b/Q_\zeta$ . Later, we illustrate through a numerical example (Table 1), how the interaction between reputation concerns and trading can lead to complex dynamics in beliefs. In fact, we show that depending on the strength of incentives to trade, and the direction of trade, beliefs can react quite differently to unexpected shocks

Given the characterization of the competitive investors beliefs in Proposition 6, we can now turn our attention to the portfolio problem of the competitive investors. The competitive investors

solve the following stochastic control problem

$$\begin{aligned} & \max_{c, q^M} \mathbb{E}_t^M \left[ \int_t^\infty e^{-r(s-t)} u_M(c_s) ds \right] \\ & \text{subject to} \\ & dW_t = (rW_t - c_t - p_t q_t^M + (\mu_D + A_0 + A_x X_t + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}}) \hat{b}_t) Y_t) dt + \sigma_D Y_t d\tilde{B}_t^D \\ & dY_t = q_t^M dt \\ & dX_t = \left( Q_0 - Q_x X_t + Q_\zeta \hat{\zeta}_t + (Q_b + Q_{\hat{b}}) \hat{b}_t \right) dt. \end{aligned}$$

Because investors do not observe  $b_t$ , the coefficients of  $\hat{b}_t$  in the law of motion of  $D_t$  and  $X_t$  given their information set is the sum of the coefficients of  $b_t$  and  $\hat{b}_t$  in the blockholder strategy. Similarly as in the baseline model, we conjecture a value function of the form

$$J(W, Y, X, \hat{b}, \hat{\zeta}) = - \frac{\exp \left( -r\gamma_M \left( W_M + H(Y, X, \hat{b}, \hat{\zeta}) \right) \right)}{r},$$

and show that the certainty equivalent  $H$  satisfies an HJB equation analogous to the one in (1). In particular, we have the following Lemma.

**Proposition 7.** *The certainty equivalent  $H$  satisfies the HJB equation*

$$\begin{aligned} rH = \max_q & (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}}) \hat{b}) Y - p(X, \hat{\zeta}, \hat{b}) q - \frac{1}{2} r\gamma_M \left[ \sigma_D^2 Y^2 + 2\sigma_D^2 \beta_D \left( H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}} \right) Y \right. \\ & \left. + \Sigma_{\hat{b}}^2 H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}}^2 H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}} \right] + \left( Q_0 - Q_x X + Q_\zeta \hat{\zeta} + (Q_b + Q_{\hat{b}}) \hat{b} \right) H_X \\ & + q H_Y - \kappa_{\hat{\zeta}} \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2} \left[ \Sigma_{\hat{b}}^2 H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}}^2 H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}} \right], \end{aligned}$$

where

$$\begin{aligned} \Sigma_{\hat{b}}^2 & \equiv \beta_q^2 \sigma_\zeta^2 + \sigma_b^2 \beta_q^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 + \beta_D^2 \sigma_D^2 \\ \Sigma_{\hat{\zeta}}^2 & \equiv \sigma_\zeta^2 \left( 1 - \frac{Q_b}{Q_\zeta} \beta_q \right)^2 + \sigma_b^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \left( 1 - \beta_q \frac{Q_b}{Q_\zeta} \right)^2 + \sigma_D^2 \beta_D^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \\ \Sigma_{\hat{b}\hat{\zeta}} & \equiv \sigma_\zeta^2 \beta_q \left( 1 - \frac{Q_b}{Q_\zeta} \beta_q \right) + \sigma_b^2 \beta_q \frac{Q_b}{Q_\zeta} \left( 1 - \beta_q \frac{Q_b}{Q_\zeta} \right) - \sigma_D^2 \beta_D^2 \frac{Q_b}{Q_\zeta} \end{aligned}$$

The coefficients  $\Sigma_{\hat{b}}^2$ ,  $\Sigma_{\hat{\zeta}}^2$  and  $\Sigma_{\hat{b}\hat{\zeta}}$  correspond to the quadratic variation and covariation of  $\hat{b}_t$  and  $\hat{\zeta}_t$ , respectively. We guess and verify that the certainty equivalent is given by a quadratic function

of the form

$$H(Y, X, \hat{b}, \hat{\zeta}) = h_0 + h_y Y + h_{\hat{\zeta}} \hat{\zeta} + h_{\hat{b}} \hat{b} + h_{xy} XY + h_{y\hat{\zeta}} \hat{\zeta} Y + h_{y\hat{b}} \hat{b} Y + h_{yy} Y^2,$$

where the coefficients are provided in equations (B.1a)-(B.1d) in the appendix. As before, taking the first order condition from the HJB equation, and invoking the market clearing condition  $X_t + Y_t = 1$ , yields the equilibrium price as given by

$$\begin{aligned} p_t &= H_Y(Y_t, X_t, \hat{b}_t, \hat{\zeta}_t) \Big|_{Y_t=1-X_t} \\ &= P_0 + P_x X_t + P_{\hat{\zeta}} \hat{\zeta}_t + P_{\hat{b}} \hat{b}_t, \end{aligned}$$

where

$$P_0 = h_y + 2h_{yy} \quad (32a)$$

$$P_x = h_{yx} - 2h_{yy} \quad (32b)$$

$$P_{\hat{\zeta}} = h_{y\hat{\zeta}} \quad (32c)$$

$$P_{\hat{b}} = h_{y\hat{b}}. \quad (32d)$$

Similar to the case without private benefit shocks, we can derive the residual supply combining the price function in (26c) with equation the equation for  $\hat{\zeta}_t$  in equation (27), and arrive to

$$\begin{aligned} R(X_t, \hat{b}_t, q_t) &= P_0 + P_x X_t + \frac{P_{\hat{\zeta}}}{Q_{\hat{\zeta}}} \left( q_t - Q_0 + Q_x X_t - (Q_{\hat{b}} + Q_b) \hat{b}_t \right) + P_{\hat{b}} \hat{b}_t \\ &= P_0 - P_{\hat{\zeta}} \frac{Q_0}{Q_{\hat{\zeta}}} + \left( P_x + P_{\hat{\zeta}} \frac{Q_x}{Q_{\hat{\zeta}}} \right) X_t + \left( P_{\hat{b}} - \frac{Q_b + Q_{\hat{b}}}{Q_{\hat{\zeta}}} P_{\hat{\zeta}} \right) \hat{b}_t + \frac{P_{\hat{\zeta}}}{Q_{\hat{\zeta}}} q_t. \end{aligned}$$

Thus, the coefficients of the residual supply are

$$R_0 = P_0 - P_{\hat{\zeta}} \frac{Q_0}{Q_{\hat{\zeta}}} \quad (33a)$$

$$R_x = P_x + P_{\hat{\zeta}} \frac{Q_x}{Q_{\hat{\zeta}}} \quad (33b)$$

$$R_{\hat{b}} = P_{\hat{b}} - \frac{Q_b + Q_{\hat{b}}}{Q_{\hat{\zeta}}} P_{\hat{\zeta}} \quad (33c)$$

$$R_q = \frac{P_{\hat{\zeta}}}{Q_{\hat{\zeta}}}. \quad (33d)$$

The last step before analyzing the blockholder's optimization problem is to analyze the evolution of the market beliefs  $\hat{b}_t$  given the blockholder's information set and given an arbitrary effort and trading strategies  $(\tilde{a}_t, \tilde{q}_t)$  that might differ to the equilibrium conjecture in (26a) and (26b).

**Lemma 4.** *Suppose that the market believes that the blockholder strategy is given by equation (26a) and (26b) but the blockholder follows the strategy  $(\tilde{q}_t, \tilde{a}_t)$ . Given the blockholder's information, the market belief  $\hat{b}_t$  follows the following stochastic differential equation*

$$d\hat{b}_t = (\mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - (\lambda - \mathcal{B}_{\hat{b}})\hat{b}_t + \mathcal{B}_q \tilde{q}_t + \beta_D \tilde{a}_t)dt + \beta_q \sigma_\zeta dB_t^\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b + \beta_D \sigma_D dB_t^D \quad (34)$$

where

$$\begin{aligned} \mathcal{B}_0 &= \beta_D (A_0 - A_\zeta \frac{Q_0}{Q_\zeta}) \\ \mathcal{B}_x &= -\beta_D \left( A_x + A_\zeta \frac{Q_x}{Q_\zeta} \right) \\ \mathcal{B}_b &= \beta_q (\kappa - \lambda) \frac{Q_b}{Q_\zeta} \\ \mathcal{B}_{\hat{b}} &= \beta_D \left( A_\zeta \frac{Q_{\hat{b}} - Q_b}{Q_\zeta} + A_b + A_{\hat{b}} \right) - \mathcal{B}_b \\ \mathcal{B}_q &= \frac{A_\zeta}{Q_\zeta} \end{aligned}$$

We can now write the blockholder problem as follows

$$\max_{c, q^L, a} \mathbb{E}_t^L \left[ \int_t^\infty e^{-r(s-t)} u_L(c_s) ds \right]$$

subject to

$$dW_t = (rW_t - c_t - \Phi(a_t, \zeta_t) - R(X_t, \hat{b}_t, q_t^L)q_t^L + (\mu_D + a_t + \delta b_t)X_t)dt + X_t \sigma_D dB_t^D$$

$$d\hat{b}_t = (\mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - (\lambda - \mathcal{B}_{\hat{b}})\hat{b}_t + \mathcal{B}_q q_t^L + \beta_D a_t)dt + \beta_q \sigma_\zeta dB_t^\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b + \beta_D \sigma_D dB_t^D$$

$$dX_t = q_t^L dt$$

We guess and verify that the the value function takes the exponential form

$$V(W, X, \zeta, b, \hat{b}) = -\frac{\exp\left(-r\gamma_L\left(W + G(X, \zeta, b, \hat{b})\right)\right)}{r}.$$

The certainty equivalent  $G$  satisfies an HJB equation similar to the one as in the case without liquidity shocks. The next lemma provides the HJB equation for  $G$ .

**Proposition 8.** *The certainty equivalent  $G$  satisfies the HJB equation*

$$\begin{aligned} rG = & \max_{a,q} (\mu_D + a + \delta b)X - R(X, \hat{b}, q)q - \Phi(a, \zeta) \\ & - \frac{r\gamma_L}{2} \left[ \sigma_D^2 X^2 + 2\sigma_D^2 \beta_D G_{\hat{b}} X + \Sigma_{\hat{b}} G_{\hat{b}}^2 + \sigma_b^2 G_b^2 + \sigma_\zeta G_\zeta^2 + \beta_q \sigma_\zeta^2 G_{\hat{b}} G_\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b^2 G_{\hat{b}} G_b \right] \\ & - \kappa \zeta G_\zeta - \lambda b G_b + \left( \mathcal{B}_0 + \mathcal{B}_x X_t + \mathcal{B}_b b_t - (\lambda - \mathcal{B}_{\hat{b}}) \hat{b}_t + \mathcal{B}_q q + \beta_D a \right) G_{\hat{b}} \\ & + q G_X + \frac{1}{2} \left[ \Sigma_{\hat{b}} G_{\hat{b}\hat{b}} + \sigma_b^2 G_{bb} + \sigma_\zeta G_{\zeta\zeta} + \beta_q \sigma_\zeta^2 G_{\hat{b}\zeta} + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b^2 G_{\hat{b}b} \right] \end{aligned}$$

It can be verified that the certainty equivalent  $G$  is given by a linear quadratic function of the form

$$\begin{aligned} G(X, \zeta, b, \hat{b}) = & g_0 + g_x X + g_\zeta \zeta + g_b b + g_{\hat{b}} \hat{b} + g_{x\zeta} \zeta X + g_{xb} b X + g_{x\hat{b}} \hat{b} X + g_{\zeta b} \zeta b \\ & + g_{\zeta \hat{b}} \zeta \hat{b} + g_{b\hat{b}} b \hat{b} + g_{xx} X^2 + g_{\zeta\zeta} \zeta^2 + g_{bb} b^2 + g_{\hat{b}\hat{b}} \hat{b}^2. \quad (35) \end{aligned}$$

The system of equations satisfied by the coefficients can be found in Section B.1 in the appendix. Taking the first order condition in the HJB equation we get that the effort and trading strategies are given by

$$a_t = \frac{\psi \zeta_t + X_t + \beta_D G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t)}{2\phi} \quad (36a)$$

$$q_t^L = \frac{G_x(X_t, \zeta_t, b_t, \hat{b}_t) + \mathcal{B}_q G_{\hat{b}}(X_t, \zeta_t, b_t, \hat{b}_t) - R_0 - R_x X_t - R_{\hat{b}} \hat{b}_t}{2R_q}. \quad (36b)$$

Equation (36a) reveals a fundamental difference between the baseline model and the model with private benefits. Under the baseline model, effort is myopic because cash flows do not provide incremental information about ability, relative to the order flow. By contrast, with private benefits, effort is forward looking. Effort has long-term implications because, by altering the cash flow, the blockholder's effort affects the market belief about ability hence the price the blockholder pays

on his upcoming trades. In this context, a positive shock to private benefits may induce the blockholder to lower his effort in order to depress the firm cash flows and manipulate the market beliefs downward. Indeed, the blockholder has an incentive to depress cash flows so the market interprets his buying new shares as driven by greater private benefits rather than higher ability. This effect can be seen clearly by looking at the first order condition of effort in equation (36a). The effect of  $\hat{b}_t$  on  $a_t$  is driven by the term  $\beta_D G_{\hat{b}}$ , which captures the effect that dividend shocks have on the market beliefs, and the impact that these beliefs have on the certainty equivalent. We can provide some intuition about the effect of reputation on effort and trading using the following representation for the equilibrium strategies

**Proposition 9.** *The equilibrium effort  $a_t$  satisfies*

$$a_t = \frac{\psi \zeta_t + X_t}{2\phi} + \frac{\beta_D}{2\phi} \mathbb{E}_t^L \left[ \int_t^\infty e^{-(r+\lambda-\mathcal{B}_{\hat{b}})(s-t)} \frac{u'_L(c_s^L)}{u'_L(c_t^L)} R_{\hat{b}} q_s^L ds \right]$$

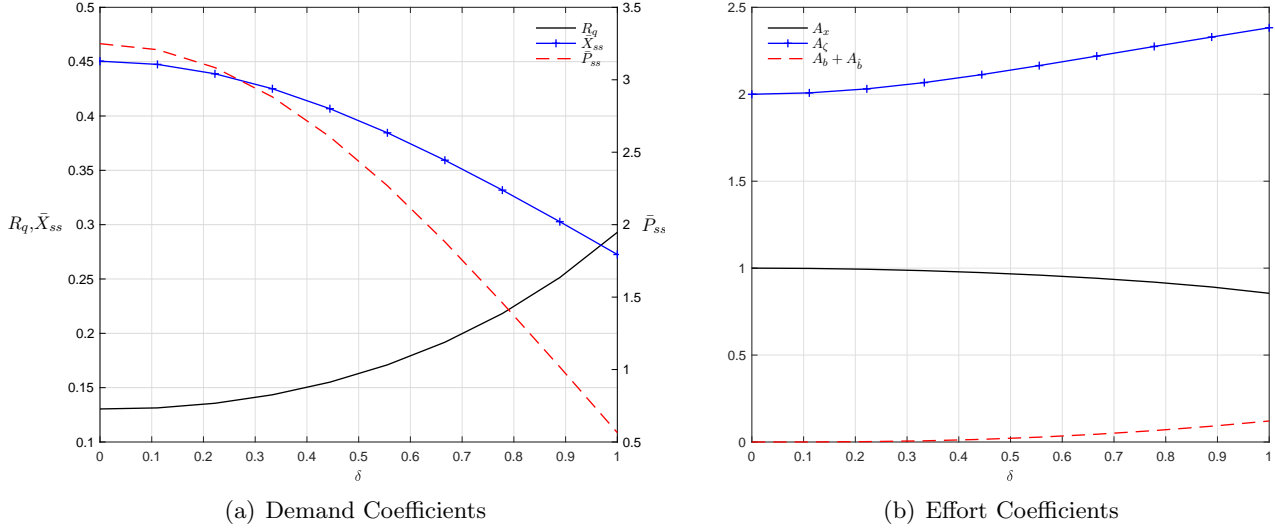
*Proof.* The proof follows from applying the envelope theorem to the HJB equation for the  $V$  and using the representation for the solution of PDE provided by the Feynman-Kack formula (Karatzas and Shreve, 2012). ■

The first term corresponds to the optimal effort in the fully revealing equilibrium, while the second term captures the impact of reputation concerns. This expression shows that the incentives to exert effort is determined by the impact of beliefs in the future residual supply faced by the blockholder, weighted by the blockholder's stochastic discount factor. This effect is discounted at  $\lambda - \mathcal{B}_{\hat{b}}$ , which captures the mean reversion of beliefs under the blockholder's information set  $\mathcal{F}_t^L$ .

In order to find the equilibrium, we need to solve a system of 15 polynomial equations, which can be found in Section B.1 in the appendix. A linear equilibrium with smooth trading exists, if this system of equations admits a solution with  $Q_x > 0$  and  $R_q > 0$ , where  $R_q$  is given in (33).

Figure 6 shows how the importance of private benefits  $\delta$  affects market liquidity  $1/R_q$  and the blockholder's steady state holdings  $\bar{X}_{ss}$ . Notice that, unlike in models with noise traders and exogenous cash-flows, price impact might increase as private benefits become more relevant. Hence, the market is less liquid even though the order flow is more noisy regarding productivity the market. This happens because private benefits are persistent so the expected blockholder position in the future is affected by the market expectation about current private benefits. If the market expects that the blockholder will buy in the future due to higher private benefits, the price will adjust upward because the larger block alleviates the moral hazard and reduces risk premium. In some cases, both  $\hat{\zeta}_t$  and  $\hat{b}_t$  increase with the order flow  $q_t$  and this magnifies the price impact (higher  $R_q$ ).





**Figure 6:** Comparative statics with respect to  $\delta$ . Parameters:  $\gamma_M = 1$ ,  $\gamma_L = 10$ ,  $\sigma_D = 1.5$ ,  $\sigma_\zeta = 1$ ,  $\sigma_b = 2$ ,  $\kappa = 1$ ,  $\lambda = 1$ ,  $\phi = 0.5$ ,  $\psi = 2$ ,  $r = 0.15$ ,  $\mu_D = 1$ .

Because it is not possible to solve this system in closed form, we analyze the dynamics of effort and trading by looking at two numerical examples. The objective of these numerical examples is not to suggest any general comparative static or prediction. Given the complexity of the incentives that arise due to the interaction between trading and effort, different qualitative patterns can arise depending on the relative magnitude of the different parameters. Instead, the objective of these examples is to illustrate qualitatively different dynamics that can arise and the economic mechanism behind them. Table 1 presents the coefficients of the equilibrium in our two leading numerical examples. The parameters in both examples are the same with the exception of the mean reversion of private benefits shocks ( $\lambda$ ). In the first example, private benefit shocks are persistent ( $\lambda = 0.3$ ) while in the second example these shocks are transitory ( $\lambda = 1.75$ ).

Figures 7 and 8 present the impulse response function for dividend shocks. We have decided to focus our attention on dividend shocks because these shocks highlight one of the main differences with the baseline model without private benefits. Without private benefits, dividends are uninformative; since trading is fully revealing, the market has nothing else to learn from the dividends. Given that dividends are i.i.d, unexpected shocks to dividends has no persistent effect. However, this is no longer the case when there are unobservable private benefits. Because trading is not fully revealing, dividends provide information to the market. Hence, shocks to dividends have a long term effect because they affect the market beliefs about the blockholder's ability.

It is important to note that, because the blockholder has superior information, the impulse response function given the blockholder information set differs from that under the market's in-

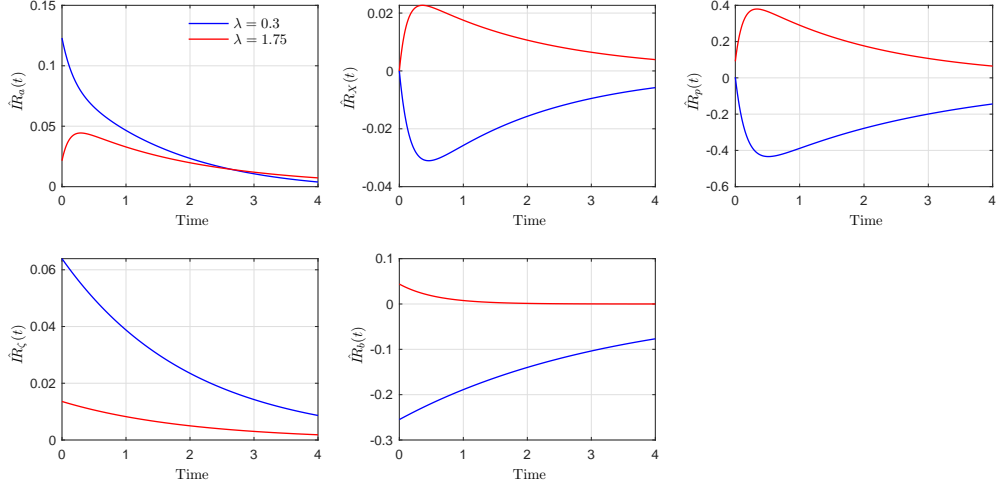
formation set. The impulse response function given the blockholder information set reflects the actual path of the equilibrium after a single shock today, while the impulse response given the information set of competitive investors captures the expected path given their limited information set. We denote the market expected impulse response by  $\hat{IR}(t)$  and the actual response by  $IR_X(t)$ . Figure 7 shows the reaction of the market expectations to an unexpected shock to dividends  $\hat{IR}(t)$ . On the one hand, when persistence is high ( $\lambda = 0.3$ ), the market attributes the shock to higher productivity ( $\beta_D$  is negative and  $Q_b/Q_\zeta$  is positive) and their expectation of the blockholder private benefits is revised downward. Accordingly, the market expects the blockholder to exert more effort but also to sell some of their shares. Even though the market expects higher productivity, the price drops because the risk premium increases (the competitive investors must absorb the of the extra shares). However, the blockholder knows that  $\zeta$  and  $b_t$  have not changed, and so knows future cash-flows are not increasing. Given this gap in expectations, the blockholder views the stock as overpriced, so he decides to sell. We can rewrite the equilibrium strategy in terms of the gap in beliefs  $\Delta_t \equiv b_t - \hat{b}_t$ . The equilibrium strategy as a function of  $\Delta_t$  is

$$q_t = Q_0 - Q_x X_t + Q_\zeta \zeta_t + (Q_b + Q_{\hat{b}}) b_t - Q_{\hat{b}} \Delta_t.$$

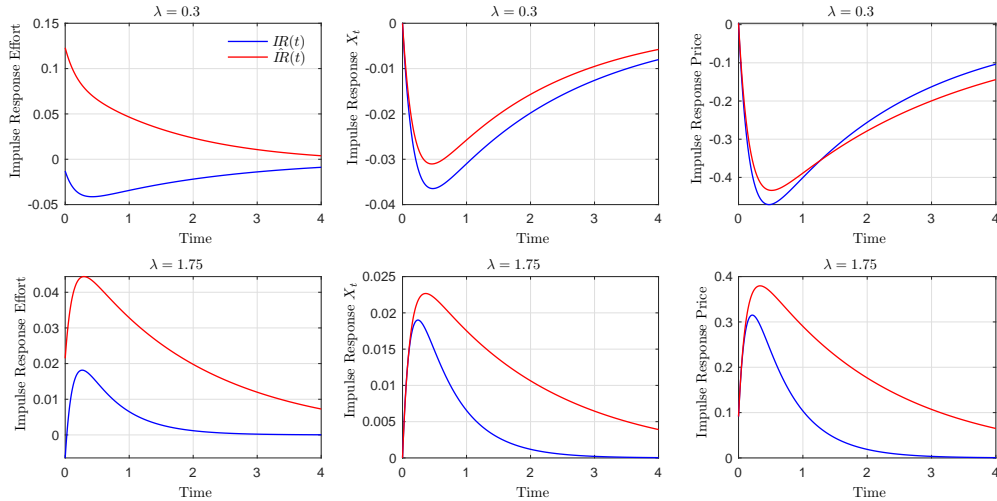
The coefficient  $Q_{\hat{b}}$  positive when  $\lambda = 0.3$ , which means that the blockholder tends to sell when the market overestimate his private benefits of holding the stock. Figure 8 shows that while the market's expectation about future effort goes up, due to the higher expected productivity, the realized effort goes down as productivity has not changed and the blockholder has weaker incentives to exert effort (because  $X_t$  goes down). Moreover, the blockholder sells even more than the market expects, which means that the price drops even more than what the market expected. The blockholder realizes a profit as he buys back the shares after the original drop in price.

We observe a very different reaction when shocks to private benefits are transitory. In this case, a positive shock to dividends is interpreted as a signal that both productivity as well as private benefits have increased. Hence, the market expects both effort as well as the block size to go up. This leads to an increment in price due to the increase in expected cash flows as well as the reduction in risk premium. The market overestimates productivity and the block size, which in this case means that the price over-shoots. While in the case of persistent shocks the price under-reacts to the shocks, the price under-shoots when the shock is transitory. Once again, the block holder is able to realize a trading profit by anticipating the overshooting.

The above results show that the market may incorrectly attribute a transitory dividend shock to variation in the blockholder's ability and private benefits, which are persistent. The blockholder observes the dividend shock and hence does not make such mistake. However, this gap between the



**Figure 7:** Impulse response for shocks to dividends. Parameters:  $\gamma_M = 1$ ,  $\gamma_L = 5$ ,  $\sigma_D = 1$ ,  $\sigma_{\zeta} = 0.5$ ,  $\sigma_b = 0.5$ ,  $\kappa = 0.5$ ,  $\phi = 0.5$ ,  $\psi = 2$ ,  $r = 0.15$ ,  $\mu_D = 1$ .



**Figure 8:** Impulse response of market expectations and blockholder reactions to an unexpected shock to dividends. Parameters:  $\gamma_M = 1$ ,  $\gamma_L = 5$ ,  $\sigma_D = 1$ ,  $\sigma_{\zeta} = 0.5$ ,  $\sigma_b = 0.5$ ,  $\kappa = 0.5$ ,  $\phi = 0.5$ ,  $\psi = 2$ ,  $r = 0.15$ ,  $\mu_D = 1$ .

market and the blockholder beliefs generates trading and also alters the agent's underlying effort. In addition, the incentives to exert effort are also distorted by the attempt to manipulate beliefs, and so prices move in the opposite direction of the blockholder's trade. If we write the equilibrium effort strategy in terms of the gap in beliefs  $\Delta_t$  we get

$$a_t = A_0 - A_x X_t + A_{\zeta} \zeta_t + (A_b + A_{\hat{j}}) b_t - A_{\hat{j}} \Delta_t.$$

By comparing our two examples, we can notice the reaction to increments in  $\Delta_t$  is linked to the reaction of both beliefs and trading to shocks. When  $\lambda = 0.3$ ,  $A_{\hat{b}}$  is positive, which means that effort is decreasing in  $\Delta_t$  leads to lower effort. Figure 7 shows how an unexpected dividend shock leads to an increment in  $\Delta_t$ . The blockholder has incentives to under provide effort to manipulate beliefs downward, decreasing price at which he will buy back the shares even further. On the other hand,  $A_{\hat{b}}$  is positive when  $\lambda = 1.75$  which means that effort is increasing in  $\Delta_t$ . However, in this case an unexpected shock in dividends lead to negative shock to  $\Delta_t$ . Hence, once again, the blockholder under-provides effort, reducing the price at which he increases his block.

In sum, Figure 8 shows the gap between market expectations and realizations of both effort and holdings. With high (low) persistence the blockholder's holdings decrease (increase). However, in both cases, the dividend shock leads the market to overestimate future effort. By contrast, the market under-estimates sales with high persistence, and over-estimates purchases with low persistence. These two examples illustrate how the effort change is driven by the Ratchet effect, that is, the blockholders incentive to manipulate future prices, by distorting current cashflows, to affect favorably future terms of trade.

$\lambda$	Effort		Trading			Beliefs and Demand		
	0.3	1.75	$Q_0$	0.3	1.75	$\beta_D$	0.3	1.75
$A_0$	0.16	-0.18	$Q_0$	4.19	7.93	$\beta_D$	-0.25	0.04
$A_x$	0.84	1.17	$Q_x$	5.87	8.26	$\beta_q$	0.32	-0.18
$A_\zeta$	2.67	1.44	$Q_\zeta$	11.58	2.20	$R_x$	16.34	15.91
$A_b$	0.13	0.18	$Q_b$	2.90	-6.85	$R_{\hat{b}}$	-0.60	0.37
$A_{\hat{b}}$	0.05	-0.14	$Q_{\hat{b}}$	1.04	5.38	$R_q$	0.56	0.31

**Table 1:** Coefficients Equilibrium. Parameters:  $\gamma_M = 1$ ,  $\gamma_L = 5$ ,  $\sigma_D = 1$ ,  $\sigma_\zeta = 0.5$ ,  $\sigma_b = 0.5$ ,  $\kappa = 0.5$ ,  $\phi = 0.5$ ,  $\psi = 2$ ,  $r = 0.15$ ,  $\mu_D = 1$ .

## 7 Empirical Implications

Evidence on ownership structure

Evidence on volatility of cash flows

Evidence re-risk premium

Evidence on the distribution of prices

**Liquidity/Effect of disclosure** In our baseline setting, the blockholder affects both the firm’s productivity and the volatility of cash flows. As the blockholder trades, he also affects the cost of capital. While on average the blockholder increases the firm productivity ( $a_t > 0$ ) he may sometimes reduce it. Since effort is random, activism induces cash flow volatility which in turn may lead to a higher risk premium.

We study the effect of uncertainty about blockholder ability. Our analysis reveals that information asymmetry about blockholder ability leads to a more concentrated ownership structure, higher firm productivity, and more volatile cash flows, but it also leads to a higher risk-premium. The price effect of information asymmetry is ambiguous.

The empirical literature has focused on understanding the price effects of activism. Prior studies find equal-weighted returns ranging from 3.4% to 7% in the days around the 13-D filing, reaching up to 11% over one or two years (Denes et al. (2017)). These results support the inference that activist interventions improve long-term value for the average firm. Klein and Zur (2009) find that hedge fund targets earn 10.2% average abnormal stock returns during the period surrounding the initial Schedule 13D. DeHaan et al. 2018 find that on a value-weighted basis, pre-to-post activism long-term returns are insignificantly different from zero.

Denes et al. (2017) finds that 8 of 11 studies on hedge fund activism conclude that earnings-based measures of operating performance improve after activist interventions, and the remaining three find no change. Brav et al. (2008) finds that activism target factories that experience abnormal declines in productivity in the years preceding the activist intervention, followed by productivity increases afterward.

Empirical studies usually measure the impact of blockholders on the stock price, which measures the value to the marginal investor (competitive fringe) but it ignores the private benefit and monitoring cost of the blockholder. Multiple studies find little evidence of an impact of blockholder on firm performance. Holderness and Sheehan (1988) finds that diffuse ownership makes no difference for Tobin’s Q.

On the other hand, Cronqvist and Fahlenbrach (2008) finds significant evidence of a positive outcome of outside blockholders. This effect is particularly strong for activist investors, pension funds and corporations. Sautner and Starks (2016) documents the different mechanisms through which blockholders intervene. Brav et al. (2008) find that activist hedge funds target firms with high operating cash flows, high return on assets, low payout and high executive compensation. These are firms that might have a more severe agency costs due to free cash flows and so benefit from monitoring. They also find evidence that activist target small companies where it is easier to acquire a significant percentage.

One reason it’s empirically difficult to evaluate blockholder’s capacity to add value, is that there

is significant heterogeneity in terms of blockholders' ability (see [Cronqvist and Fahlenbrach \(2008\)](#)). Second, investor activism affects firm value via multiple channels and it's difficult to separate each of them empirically.

## 7.1 Summary Empirical Literature

Some notes on empirical implications.

1. Barclay and Holderness (1991) find that trade of large blocks between investors lead to 16% increase in market value. [Holderness and Sheehan \(1988\)](#) show that trades of majority blocks also lead to increments in price. Empirical studies usually measure the impact of blockholders in the price, which measures the value to the marginal investor (competitive fringe) and ignores the private benefit and monitoring cost of the blockholder.
2. Multiple authors find little evidence of an impact of blockholder on firm performance. For example, Barclay and Holderness (1989) find that the average private benefit is 4.3% of the total value of equity. On a similar line, [Holderness and Sheehan \(1988\)](#) finds that diffuse ownership makes no difference for Tobin's Q.
3. The evidence of blockholder is stronger in countries with low investor protection. Studies find that Tobin's Q is positively related to the decision rights of non-management blockholders.
4. On the other hand, [Cronqvist and Fahlenbrach \(2008\)](#) finds significant evidence of a positive outcome of outside blockholders. This effect is particularly strong for activist investors, pension funds and corporations.
5. McCahery et al. (2016) documents the different mechanisms through which blockholders intervene (check more).
6. Brav et al. (2008) find that activist hedge funds focus on firms with high operating cash flows, high return on assets, low payout and high executive compensation. These are firms that might have a more severe agency cost due to free cash flows and so benefit from monitoring. They also find evidence that activist target small companies where it is easier to acquire a significant percentage.
7. Gantchev and Jotikasthira (2018) finds evidence that hedge fund activists camouflage purchases by timing them to coincide with liquidity-driven selling by institutions.

## 8 Conclusion

This paper studies strategic trading and activism. We propose a model where a blockholder affects the firm value via trading and activism. We consider two regimes, one where the blockholder's ability to add value is observed by the market and another where only the blockholder observes his ability. The presence of information asymmetry about ability fundamentally alters the dynamics of firm value and its long-run level.

Without information asymmetry, the blockholder's trading is characterized by Coasian dynamics; that is, the blockholder always trades at a competitive price. In this context, an improvement in the blockholder's ability to add value increases the blockholder's exposure to the firm cash flows, and thus induces the blockholder to reduce his holdings. By doing so, the blockholder effectively deprives other shareholders from some of the potential benefits of his activism.

The blockholder's behavior drastically changes under information asymmetry. When the market does not observe blockholder ability, it infers ability based on the blockholder's trading. The blockholder must trade gradually to mitigate the price impact of his order flow. Furthermore, the blockholder responds to a positive ability shock by buying shares. As the blockholder stake grows, his activism incentives become stronger and the firm's productivity goes up. Also, as the blockholder holdings grow, the risk premium decreases, since the blockholder absorbs a greater fraction of the firm's cash flow risk.

Our results suggest the information asymmetry not only affects the dynamics of activism and asset pricing, but also the long-run outcome. Notably, the presence of information asymmetry modifies the firm's ownership structure causing the blockholder to hold a larger stake. This effect is particularly strong when the blockholder's ability is more relevant. It survives even in the absence of moral hazard, but does require risk aversion by the market.

The literature has focused on whether/how liquidity interacts with activism ([Maug \(1998\)](#)). In our setting, the presence of information asymmetry reduces market's liquidity in that the order flow has a price impact under information asymmetry. Such illiquidity effect has some positive effects, insofar as it induces the blockholder to hold a larger stake. As such, the information asymmetry restores the incentive of the blockholder to hold a large relatively but undiversified portfolio. In the long-run, this leads to more activism, and a higher firm productivity, but it also exacerbates the cash flow volatility, causing a higher risk premium.

Our model has a number of limitations. First, we model activism as having only short-term effects but, in practice, activism has persistent effects on the firm's cash flows. Relaxing this assumption would be useful if one wishes to understand how policy makers should address blockholder's myopia, namely the blockholder's tendency to underestimate the long-run consequences of

their interventions (cites ). Second, we assume a blockholder holdings are observable. In practice, their holdings are observed with some delay.<sup>10</sup> Third, our model captures the interventions of a blockholder in a stationary environment where the average holdings of the blockholder are positive. By contrast, the interventions of hedge-fund activists take place over a limited period of time; they are not meant to last forever.

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<sup>10</sup>The Williams Act of 1968 requires that investors must disclose ownership stakes of more than 5% within 10 days. In Britain investors must disclose stakes of more than 5% within two days.



## References

- Admati, A. R. and P. Pfleiderer (2009). *The Review of Financial Studies* 22(7), 2645–2685.
- Admati, A. R., P. Pfleiderer, and J. Zechner (1994). Large shareholder activism, risk sharing, and financial market equilibrium. *Journal of Political Economy* 102(6), 1097–1130.
- Albuquerque, R. and N. Wang (2008). Agency conflicts, investment, and asset pricing. *The Journal of Finance* 63(1), 1–40.
- Back, K., P. Collin-Dufresne, V. Fos, T. Li, and A. Ljungqvist (2018). Activism, strategic trading, and liquidity. *Econometrica* 86(4), 1431–1463.
- Becht, M., P. Bolton, and A. Röell (2003). Corporate governance and control. In *Handbook of the Economics of Finance*, Volume 1, pp. 1–109. Elsevier.
- Brav, A., W. Jiang, F. Partnoy, and R. Thomas (2008). Hedge fund activism, corporate governance, and firm performance. *The Journal of Finance* 63(4), 1729–1775.
- Campbell, J. Y. and A. S. Kyle (1993). Smart money, noise trading and stock price behaviour. *The Review of Economic Studies* 60(1), 1–34.
- Coase, R. H. (1972). Durability and monopoly. *The Journal of Law and Economics* 15(1), 143–149.
- Coffee, J. C. (1991). Liquidity versus control: The institutional investor as corporate monitor. *Columbia Law Review* 91(6), 1277–1368.
- Cronqvist, H. and R. Fahlenbrach (2008). Large shareholders and corporate policies. *The Review of Financial Studies* 22(10), 3941–3976.
- Dávila, E. and C. Parlatore (2017). Trading costs and informational efficiency.
- DeMarzo, P. M. and B. Urošević (2006). Ownership dynamics and asset pricing with a large shareholder. *Journal of Political Economy* 114(4), 774–815.
- Denes, M. R., J. M. Karpoff, and V. B. McWilliams (2017). Thirty years of shareholder activism: A survey of empirical research. *Journal of Corporate Finance* 44, 405–424.
- Dou, Y., O.-K. Hope, W. B. Thomas, and Y. Zou (2016). Individual large shareholders, earnings management, and capital-market consequences. *Journal of Business Finance & Accounting* 43(7-8), 872–902.

- Edmans, A. (2009). Blockholder trading, market efficiency, and managerial myopia. *The Journal of Finance* 64(6), 2481–2513.
- Edmans, A., I. Goldstein, and W. Jiang (2015). Feedback effects, asymmetric trading, and the limits to arbitrage. *American Economic Review* 105(12), 3766–97.
- Edmans, A. and C. G. Holderness (2017). Blockholders: A survey of theory and evidence.
- Evans, L. C. (2012). *An introduction to stochastic differential equations*, Volume 82. American Mathematical Soc.
- Faure-Grimaud, A. and D. Gromb (2004). Public trading and private incentives. *Review of Financial Studies* 17(4), 985–1014.
- Gantchev, N. and C. Jotikasthira (2018). Institutional trading and hedge fund activism. *Management Science* 64(6), 2930–2950.
- Gomes, A. (2000). Going public without governance: Managerial reputation effects. *Journal of Finance* 55(2), 615–646.
- Gorton, G. B., P. He, and L. Huang (2014). Agency-based asset pricing. *Journal of Economic Theory* 149, 311–349.
- Grossman, S. J. and O. D. Hart (1980). Takeover bids, the free-rider problem, and the theory of the corporation. *The Bell Journal of Economics*, 42–64.
- Hatchondo, J., P. Krusell, and M. Schneider (2014). Asset trading and valuation with uncertain exposure.
- Hirschman, A. O. (1970). *Exit, voice, and loyalty: Responses to decline in firms, organizations, and states*, Volume 25. Harvard university press.
- Holderness, C. G. and D. P. Sheehan (1988). The role of majority shareholders in publicly held corporations: An exploratory analysis. *Journal of financial economics* 20, 317–346.
- Holmstrom, B. and P. Milgrom (1987). Aggregation and linearity in the provision of intertemporal incentives. *Econometrica: Journal of the Econometric Society*, 303–328.
- Huddart, S. (1993). The effect of a large shareholder on corporate value. *Management Science* 39(11), 1407–1421.

- Kahn, C. and A. Winton (1998). Ownership structure, speculation, and shareholder intervention. *Journal of Finance* 53(1), 99–129.
- Karatzas, I. and S. Shreve (2012). *Brownian Motion and Stochastic Calculus*, Volume 113. Springer.
- Kihlstrom, R. (2000). Monopoly power in dynamic securities markets.
- Klein, A. and E. Zur (2009). Entrepreneurial shareholder activism: Hedge funds and other private investors. *The Journal of Finance* 64(1), 187–229.
- Kubler, F., P. Renner, and K. Schmedders (2014). Computing all solutions to polynomial equations in economics. In *Handbook of Computational Economics*, Volume 3, pp. 599–652. Elsevier.
- Kunita, H. (1993). Representation and stability of nonlinear filters associated with gaussian noises. In *Stochastic processes*, pp. 201–210. Springer.
- Kyle, A. S. (1985). Continuous auctions and insider trading. *Econometrica*, 1315–1335.
- Kyle, A. S., A. A. Obizhaeva, and Y. Wang (2017). Smooth trading with overconfidence and market power. *Review of Economic Studies* 85(1), 611–662.
- Kyle, A. S. and J.-L. Vila (1991). Noise trading and takeovers. *The RAND Journal of Economics*, 54–71.
- Leland, H. E. and D. H. Pyle (1977). Informational asymmetries, financial structure, and financial intermediation. *Journal of Finance* 32(2), 371–387.
- Liptser, R. S. and A. N. Shiryaev (2001b). *Statistics of Random Processes I. General Theory* (2 ed.), Volume 5. Springer-Verlag Berlin Heidelberg.
- Liptser, R. S. and A. N. Shiryaev (2001a). *Statistics of Random Processes II. Applications* (2 ed.), Volume 5. Springer-Verlag Berlin Heidelberg.
- Liu, Z. and J. Xiong (2010). Some solvable classes of filtering problem with Ornstein-Uhlenbeck noise. *Communications on Stochastic Analysis* 4(1), 4.
- Manzano, C. and X. Vives (2011). Public and private learning from prices, strategic substitutability and complementarity, and equilibrium multiplicity. *Journal of Mathematical Economics* 47(3), 346–369.
- Maug, E. (1998). Large shareholders as monitors: is there a trade-off between liquidity and control? *Journal of Finance* 53(1), 65–98.

- McCahery, J. A., Z. Sautner, and L. T. Starks (2016). Behind the scenes: The corporate governance preferences of institutional investors. *The Journal of Finance* 71(6), 2905–2932.
- Noe, T. H. (2002). Investor activism and financial market structure. *The Review of Financial Studies* 15(1), 289–318.
- Shleifer, A. and R. W. Vishny (1986). Large shareholders and corporate control. *Journal of political economy* 94(3, Part 1), 461–488.
- Vayanos, D. (1999). Strategic trading and welfare in a dynamic market. *Review of Economic Studies* 66(2), 219–254.
- Vayanos, D. (2001). Strategic trading in a dynamic noisy market. *Journal of Finance* 56(1), 131–171.
- Vayanos, D. and P. Woolley (2013). An institutional theory of momentum and reversal. *Review of Financial Studies* 26(5), 1087–1145.
- Xiong, J. (2008). *An Introduction to Stochastic Filtering Theory*, Volume 18. Oxford University Press.

## A Appendix

### Proof Lemma 1

*Proof.* The HJB equation is

$$rJ = \max_{c,q} u_M(c) + \left( rW - c - p(X, \hat{\zeta})q + Y(\mu_D + A_x X + A_\zeta \hat{\zeta}) \right) J_W \\ + qJ_Y + (Q_0 - Q_x X + Q_\zeta \hat{\zeta})J_X - \kappa \hat{\zeta} J_\zeta + \frac{1}{2} (Y^2 \sigma_D^2 J_{WW} + \sigma_\zeta^2 J_{\zeta\zeta})$$

The first condition for the consumption choice is

$$\frac{\partial u_M(c)}{\partial c} = J_W,$$

and using our conjectured value function  $J$  we get

$$u_M(c) = rJ \\ c = r(W + H(Y, \hat{\zeta}, X))$$

Replacing in the HJB equation

$$rH = \max_q (\mu_D + A_x X + A_\zeta \hat{\zeta})Y - p(X, \hat{\zeta})q - \frac{1}{2} (r\gamma_M Y^2 \sigma_D^2 + r\gamma_M \sigma_\zeta^2 H_\zeta^2) \\ qH_Y + (Q_0 - Q_x X + Q_\zeta \hat{\zeta})H_X - \kappa \hat{\zeta} H_\zeta + \frac{1}{2} \sigma_\zeta^2 H_{\zeta\zeta}$$

We conjecture a quadratic form for the certainty equivalent  $H$

$$H(Y, \hat{\zeta}, X) = h_y Y + h_{yx} XY + h_{y\zeta} Y \hat{\zeta} + h_{yy} Y^2$$

Replacing, we get

$$r(h_y Y + h_{yx} XY + h_{y\zeta} Y \hat{\zeta} + h_{yy} Y^2) = Y(\mu_D + A_x X + A_\zeta \hat{\zeta}) \\ + (Q_0 - Q_x X + Q_\zeta \hat{\zeta})h_{yx} Y - \kappa h_{y\zeta} Y \hat{\zeta} - \frac{r\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2) Y^2$$

We get the following system of equations

$$\begin{aligned}
rh_y &= \mu_D + Q_0 h_{yx} \\
rh_{yx} &= A_x - Q_x h_{yx} \\
rh_{y\zeta} &= A_\zeta + Q_\zeta h_{yx} - \kappa h_{y\zeta} \\
rh_{yy} &= -\frac{r\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2)
\end{aligned}$$

Solving the system we get

$$\begin{aligned}
h_y &= \frac{\mu_D}{r} + \frac{A_x Q_0}{r(r + Q_x)} \\
h_{yx} &= \frac{A_x}{r + Q_x} \\
h_{y\zeta} &= \frac{1}{r + \kappa} \left( A_\zeta + \frac{A_x Q_\zeta}{r + Q_x} \right) \\
h_{yy} &= -\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2).
\end{aligned}$$

■

### Proof Proposition 1

*Proof.* The first condition (2) implies that

$$p_t = H_Y(Y_t, \hat{\zeta}_t, X_t)$$

Using the envelope condition in equation 1, we get

$$\begin{aligned}
rH_Y &= \mu_D + A_x X + A_\zeta \hat{\zeta} - r\gamma_M (Y\sigma_D^2 + \sigma_\zeta^2 H_\zeta H_{Y\zeta}) \\
&\quad + (Q_0 - Q_x X + Q_\zeta \hat{\zeta}) H_{YX} - \kappa \hat{\zeta} H_{Y\zeta} + \frac{1}{2} \sigma_\zeta^2 H_{\zeta\zeta Y} \quad (\text{A.1})
\end{aligned}$$

Replacing  $H_Y = p$  and we get

$$rp = \mu_D + A_x X + A_\zeta \hat{\zeta} - r\gamma_M (Y\sigma_D^2 + \sigma_\zeta^2 H_\zeta H_{Y\zeta}) + (Q_0 - Q_x X + Q_\zeta \hat{\zeta}) p_X - \kappa \hat{\zeta} p_\zeta + \frac{1}{2} \sigma_\zeta^2 p_{\zeta\zeta}$$

Using the Feynman-Kac formula (Karatzas and Shreve, 2012), we get the following stochastic representation for the solution of equation (A.2).

$$p_t = \mathbb{E}_t^M \left[ \int_t^\infty e^{-r(s-t)} \left( \mu_D + A_x X_s + A_\zeta \hat{\zeta}_s - r\gamma_M \left( \sigma_D^2 Y_s + \sigma_\zeta^2 H_\zeta H_{Y_\zeta}(Y_s, \hat{\zeta}_s, X_s) \right) \right) ds \right].$$

Replacing the certainty equivalent  $H$ , and imposing the market clearing condition  $X_t + Y_t = 1$ , we get

$$p_t = \mathbb{E}_t^M \left[ \int_t^\infty e^{-r(s-t)} \left( \mu_D + A_x X_s + A_\zeta \hat{\zeta}_s - r\gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 h_{y_\zeta}^2 \right) (1 - X_s) \right) ds \right].$$

Finally, we arrive to the expression in by replacing  $h_{y_\zeta}$  in Lemma 1. ■

## Proof Lemma 2

*Proof.* Replacing our conjecture for the certainty equivalent in (4), we get the following system of for the coefficients in  $G$ .

$$r g_0^o = \sigma_\zeta^2 g_{\zeta\zeta}^o \quad (\text{A.2})$$

$$r g_x^o = \mu_D \quad (\text{A.3})$$

$$(r + \kappa) g_{x\zeta}^o = \frac{\psi}{2\phi} - (2r\gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^o) g_{x\zeta}^o \quad (\text{A.4})$$

$$(r + 2\kappa) g_{\zeta\zeta}^o = -2r\gamma_L \sigma_\zeta^2 (g_{\zeta\zeta}^o)^2 + \frac{\psi^2}{4\phi} \quad (\text{A.5})$$

$$r g_{xx}^o = \frac{1}{4\phi} - \frac{r\gamma_L}{2} \left( \sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2 \right) \quad (\text{A.6})$$

From here, we immediately get that  $g_x = \mu_D/r$  and

$$g_{\zeta\zeta}^o = \frac{\pm \sqrt{(r + 2\kappa)^2 + 2r\gamma_L \sigma_\zeta^2 \frac{\psi^2}{\phi}} - (r + 2\kappa)}{4r\gamma_L \sigma_\zeta^2}$$

The rest of the expression follow directly. To verify that  $G_+(X, \zeta) > G_-(X, \zeta)$ , notice that, because  $g_{0+}^o > v = g_{0-}^o$ , we have that  $g_{0+}^o + g_{x+}^o X > g_{0-}^o + g_{x-}^o X$ , where  $g_{+}^o$  and  $g_{-}^o$  are the coefficients of  $G_+^o$  and  $G_-^o$  respectively. Next, let

$$M \equiv \begin{pmatrix} g_{\zeta\zeta+}^o - g_{\zeta\zeta-}^o & \frac{1}{2}(g_{x\zeta+}^o - g_{x\zeta-}^o) \\ \frac{1}{2}(g_{x\zeta+}^o - g_{x\zeta-}^o) & g_{xx+}^o - g_{xx-}^o \end{pmatrix}$$

be the difference in the quadratic coefficients  $G_+^o$  and  $G_-^o$ . The eigenvalues of  $M$  are 0 and

$$\frac{\sqrt{\frac{2r\sigma_\zeta^2\psi^2\gamma_L}{\phi} + (r + 2\kappa)^2 \left( r\sigma_\zeta^2\psi^2\gamma_L \left( r\sigma_\zeta^2 (\psi^2 + 1) \gamma_L + 4\kappa\phi(r + \kappa) \right) + 4\kappa^2\phi^2(r + \kappa)^2 \right)}}{2r\gamma_L \left( r\sigma_\zeta^3\psi^2\gamma_L + 2\kappa\sigma_\zeta\phi(r + \kappa) \right)^2} > 0,$$

which means that  $M$  is positive semidefinite. It follows that  $(\zeta, X)M(\zeta, X)^\top \geq 0$ , which means that  $g_{xx+}^o X^2 + g_{\zeta\zeta+}^o \zeta^2 + g_{x\zeta+}^o X\zeta \geq g_{xx-}^o X^2 + g_{\zeta\zeta-}^o \zeta^2 + g_{x\zeta-}^o X\zeta$  for all  $(X, \zeta)$ .

■

## Proof Proposition 2

*Proof.* Using the certainty equivalent for the blockholder, together with the first order condition we get that coefficients in the price function are

$$\begin{aligned} P_0 &= g_x^o \\ P_x &= 2g_{xx}^o \\ P_\zeta &= g_{x\zeta}^o. \end{aligned}$$

Moreover, from the solution of the market makers problem we also have that the coefficients are given by

$$\begin{aligned} P_0 &= h_y + 2h_{yy}S \\ P_x &= h_{yx} - 2h_{yy} \\ P_\zeta &= h_{y\zeta}, \end{aligned}$$

where

$$\begin{aligned} h_y &= \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi(r + Q_x^o)} \\ h_{yx} &= \frac{1}{2r\phi(r + Q_x^o)} \\ h_{y\zeta} &= \frac{1}{r + \kappa} \left( \frac{\psi}{2\phi} + \frac{Q_\zeta^o}{2\phi(r + Q_x^o)} \right) \\ h_{yy} &= -\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2). \end{aligned}$$



That is, in equilibrium, the marginal valuation of the large shareholder and the one of the competitive investors must coincide. Matching coefficients, we get

$$g_x^o = \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi r(r + Q_x^o)} - 2\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2) \quad (\text{A.7})$$

$$2g_{xx}^o = \frac{A_x}{r + Q_x^o} + \gamma_M (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2) \quad (\text{A.8})$$

$$g_{x\zeta}^o = \frac{1}{r + \kappa} \left( A_\zeta + \frac{A_x Q_\zeta^o}{r + Q_x^o} \right) \quad (\text{A.9})$$

We can solve for  $Q_x^o, Q_\zeta^o$  using equations (A.8) and (A.9)

$$\begin{aligned} \frac{1}{r\phi} - \gamma_L (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^{\text{obs}})^2) &= \frac{2}{2\phi(r + Q_x^o)} + \gamma_M (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2) \\ \frac{\psi}{2\phi} + \frac{Q_\zeta^o}{2\phi(r + Q_x^o)} &= (r + \kappa) g_{x\zeta}^o, \end{aligned}$$

which yields

$$\begin{aligned} Q_x^o &= \frac{r^2 \phi (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)}{1 - r\phi (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)} \\ Q_\zeta^o &= (r + Q_x^o) (2(r + \kappa)\phi (g_{x\zeta}^o)^2 - \psi) \end{aligned}$$

For  $Q_0^o$ , we use the equation

$$g_x^o = \frac{\mu_D}{r} + \frac{Q_0^o}{2\phi r(r + Q_x^o)} - 2\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2),$$

and replacing  $g_\zeta^o = 0$  and  $g_x^o = \frac{\mu_D}{r}$ , we get

$$0 = \frac{Q_0^o}{2\phi r(r + Q_x^o)} - 2\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2),$$

so

$$Q_0^o = 2r\gamma_M \phi (r + Q_x^o) (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2).$$

Replacing  $Q_x^o$ , we arrive to

$$Q_0^o = \frac{r^2 \gamma_M (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)}{(2\phi)^{-1} - r (\gamma_L + \gamma_M) (\sigma_D^2 + \sigma_\zeta^2 (g_{x\zeta}^o)^2)}.$$

■

### Proof Lemma 3

*Proof.* The derivation of the certainty equivalent for the large shareholder is similar to the one for market makers. If we conjecture the following quadratic function for the certainty equivalent

$$G(\zeta, X) = g_0 + g_x X + g_\zeta \zeta + g_{xx} X^2 + g_{\zeta\zeta} \zeta^2 + g_{x\zeta} X \zeta,$$

then we get that

$$a = \frac{\psi \zeta + X}{2\phi}$$

$$q = \frac{g_x - R_0 + (2g_{xx} - R_x)X + g_{x\zeta}\zeta}{2R_q}.$$

Replacing in the HJB equation, and matching coefficients, we arrive to the system of equations in the Lemma. ■

## A.1 Proofs

### Proof Proposition 3

*Proof.* The coefficients  $(\mathbf{A}, \mathbf{Q})$  are

$$A_x = \frac{1}{2\phi}$$

$$A_\zeta = \frac{\psi}{2\phi}$$

$$Q_0 = \frac{g_x - R_0}{2R_q}$$

$$Q_x = \frac{R_x - 2g_{xx}}{2R_q}$$

$$Q_\zeta = \frac{g_{x\zeta}}{2R_q}$$

It is convenient to express the coefficients  $\mathbf{R}$  in terms of the coefficients  $\mathbf{Q}$

$$\begin{aligned} R_0 &= \frac{Q_\zeta g_x - Q_0 g_{x\zeta}}{Q_\zeta} \\ R_q &= \frac{g_{x\zeta}}{2Q_\zeta} \\ R_x &= 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \end{aligned}$$

Replacing this in the system of equations in Lemma 2, we get the system

$$\begin{aligned} r g_0 &= -\frac{1}{2} r \gamma_L g_\zeta^2 \sigma_\zeta^2 + \frac{1}{2} \frac{Q_0^2}{Q_\zeta} g_{x\zeta} + \sigma_\zeta^2 g_{\zeta\zeta} \\ r g_x &= \mu_D - \frac{Q_0 Q_x}{Q_\zeta} g_{x\zeta} - r \gamma_L \sigma_\zeta^2 g_\zeta g_{x\zeta} \\ (r + \kappa) g_\zeta &= -2r \gamma_L \sigma_\zeta^2 g_{\zeta\zeta} g_\zeta + Q_0 g_{x\zeta} \\ (r + \kappa) g_{x\zeta} &= \frac{\psi}{2\phi} - (Q_x + 2r \gamma_L \sigma_\zeta^2 g_{\zeta\zeta}) g_{x\zeta} \\ (r + 2\kappa) g_{\zeta\zeta} &= \frac{Q_\zeta}{2} g_{x\zeta} - 2r \gamma_L \sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \\ r g_{xx} &= \frac{Q_x^2}{2Q_\zeta} g_{x\zeta} + \frac{1}{4\phi} - \frac{r \gamma_L}{2} (\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2) \end{aligned}$$

The next step is to find expressions for the coefficients  $\mathbf{Q}$ . The coefficients in the price function are

$$\begin{aligned} P_0 &= g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} \\ P_x &= 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \\ P_\zeta &= \frac{g_{x\zeta}}{2}. \end{aligned}$$

But, from the solution of the market makers' problem we have that

$$\begin{aligned} P_0 &= h_y + 2h_{yy} \\ P_x &= h_{yx} - 2h_{yy} \\ P_\zeta &= h_{y\zeta}, \end{aligned}$$

where

$$\begin{aligned}
h_y &= \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r+Q_x)} \\
h_{yx} &= \frac{1}{2\phi(r+Q_x)} \\
h_{y\zeta} &= \frac{1}{r+\kappa} \left( \frac{\psi}{2\phi} + \frac{Q_\zeta}{2\phi(r+Q_x)} \right) \\
h_{yy} &= -\frac{\gamma_M}{2} (\sigma_D^2 + \sigma_\zeta^2 h_{y\zeta}^2).
\end{aligned}$$

Matching coefficients

$$\begin{aligned}
g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} &= \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r+Q_x)} - \gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \\
2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} &= \frac{1}{2\phi(r+Q_x)} + \gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \\
g_{x\zeta} &= \frac{1}{r+\kappa} \left( \frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r+Q_x)} \right)
\end{aligned}$$

The following block of equations can be solved independently

$$(r+\kappa)g_{x\zeta} = \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta})g_{x\zeta} \quad (\text{A.10})$$

$$(r+2\kappa)g_{\zeta\zeta} = \frac{Q_\zeta}{2} g_{x\zeta} - 2r\gamma_L\sigma_\zeta^2 g_{\zeta\zeta}^2 + \frac{\psi^2}{4\phi} \quad (\text{A.11})$$

$$r g_{xx} = \frac{Q_x^2}{2Q_\zeta} g_{x\zeta} + \frac{1}{4\phi} - \frac{r\gamma_L}{2} (\sigma_D^2 + \sigma_\zeta^2 g_{x\zeta}^2) \quad (\text{A.12})$$

$$2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} = \frac{1}{2\phi(r+Q_x)} + \gamma_M \left( \sigma_D^2 + \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \quad (\text{A.13})$$

$$g_{x\zeta} = \frac{1}{r+\kappa} \left( \frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r+Q_x)} \right) \quad (\text{A.14})$$

Finally, using equations (A.12) and (A.13) to eliminate  $g_{xx}$  we arrive to the system in the proposition. Moreover, the solution is an equilibrium if the condition  $Q_x > 0$  and  $R_q > 0$  is satisfied. ■

### Proof Proposition 4

We consider the limit when  $\sigma_\zeta^\epsilon = \sqrt{\epsilon}\sigma_\zeta$  and  $\kappa^\epsilon = \epsilon\kappa$ . We need to be careful when we take the limit because the term  $\epsilon\sigma_\zeta^2$  multiplies the leading monomial in two of the equations so this is a singular perturbation problem. For this reason, we consider the scaled variable  $g_{x\zeta} = \hat{g}_{x\zeta}/\epsilon$ . Replacing in the system of equation in Proposition 3 and taking the limit when  $\epsilon \rightarrow 0$  we get

$$\begin{aligned} r g_{x\zeta} &= \frac{\psi}{2\phi} - (Q_x + 2r\gamma_L\sigma_\zeta^2\hat{g}_{x\zeta})g_{x\zeta} \\ 2r\gamma_L\sigma_\zeta^2\hat{g}_{x\zeta}^2 + r\hat{g}_{x\zeta} &= 0 \\ \frac{Q_x}{2Q_\zeta}(r + 2Q_x)g_{x\zeta} &= -\frac{Q_x}{2\phi(r + Q_x)} + r(\gamma_L + \gamma_M)\sigma_D^2 \\ r g_{x\zeta} &= \frac{\psi}{\phi} + \frac{Q_\zeta}{\phi(r + Q_x)} \end{aligned}$$

From here, we take the negative root for  $\hat{g}_{x\zeta}$

$$\hat{g}_{x\zeta} = -\frac{1}{2\gamma_L\sigma_\zeta^2} \quad (\text{A.15})$$

Replacing back in the first equation we get

$$g_{x\zeta} = \frac{\psi}{2\phi Q_x} \quad (\text{A.16})$$

Replacing in the last two equation we get

$$\frac{\psi(r + Q_x)^2}{Q_\zeta} + Q_x = 2r\sigma_D^2(\gamma_L + \gamma_M)\phi(r + Q_x) \quad (\text{A.17})$$

$$\frac{Q_\zeta}{r + Q_x} + \psi = \frac{r\psi}{2Q_x} \quad (\text{A.18})$$

Using the second equation to solve for  $Q_\zeta$  we get

$$Q_\zeta = \frac{\psi(r + Q_x)(r - 2Q_x)}{2Q_x} \quad (\text{A.19})$$

Replacing in the first equation above we get

$$r\phi(\gamma_L + \gamma_M)\sigma_D^2 = \frac{rQ_x}{(r + Q_x)(r - 2Q_x)} \quad (\text{A.20})$$

Using the definition for  $\alpha$  in the proposition, we get

$$Q_x = r \frac{\sqrt{9\alpha^2 + 2\alpha + 1} - \alpha - 1}{4\alpha}$$

and

$$Q_\zeta = \frac{r\psi}{2\alpha} \tag{A.21}$$

Moreover, using the the fact that  $R_q = g_{x\zeta}/2Q_\zeta$ , we can write the expression for  $Q_x$  in terms of  $R_q$  using equation (A.16). Similarly, we can replace  $Q_x$  and  $Q_\zeta$  in the expressions for  $P_x$  and  $P_\zeta$  and after some simple manipulations we arrive to the expressions in the proposition. Next, we derive the limit for  $\bar{X}_{ss}$ . We have the system

$$\begin{aligned} g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} &= \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r + Q_x)} - \gamma_M \left( \sigma_D^2 + \epsilon \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \\ r g_x &= \mu_D - \frac{Q_0 Q_x}{Q_\zeta} g_{x\zeta} - r \gamma_L \sigma_\zeta^2 \epsilon g_\zeta g_{x\zeta} \\ Q_0 g_{x\zeta} &= (r + \epsilon \kappa + 2r \gamma_L \sigma_\zeta^2 \epsilon g_\zeta \zeta) g_\zeta \end{aligned}$$

We can eliminate  $g_\zeta$  and get

$$\begin{aligned} g_x - \frac{Q_0}{2Q_\zeta} g_{x\zeta} &= \frac{\mu_D}{r} + \frac{Q_0}{2\phi r(r + Q_x)} - \gamma_M \left( \sigma_D^2 + \epsilon \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right) \\ g_x &= \frac{\mu_D}{r} - \frac{Q_0 Q_x}{r Q_\zeta} g_{x\zeta} - \gamma_L \sigma_\zeta^2 \frac{\epsilon Q_0 g_{x\zeta}^2}{r + \epsilon \kappa + 2r \gamma_L \sigma_\zeta^2 \epsilon g_\zeta \zeta}. \end{aligned}$$

From here, we get the equation

$$-\frac{Q_0 Q_x}{r Q_\zeta} g_{x\zeta} - \frac{\epsilon \gamma_L \sigma_\zeta^2}{r + \epsilon \kappa + 2r \gamma_L \sigma_\zeta^2 \epsilon g_\zeta \zeta} Q_0 g_{x\zeta}^2 = \frac{Q_0}{2Q_\zeta} g_{x\zeta} + \frac{Q_0}{2\phi r(r + Q_x)} - \gamma_M \left( \sigma_D^2 + \epsilon \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right)$$

Notice that from Proposition 3 we have that  $g_{\zeta\zeta}$  satisfies the following equation

$$2r \gamma_L \sigma_\zeta^2 \epsilon g_{\zeta\zeta} = \frac{\epsilon \frac{Q_\zeta}{2} g_{x\zeta} + \epsilon \frac{\psi^2}{4\phi}}{\epsilon g_{\zeta\zeta}} - (r + 2\epsilon \kappa),$$

which means that we can write

$$-\frac{Q_0 Q_x}{r Q_\zeta} g_{x\zeta} - \frac{2\gamma_L \sigma_\zeta^2 \hat{g}_{\zeta\zeta}}{Q_\zeta g_{x\zeta} + \frac{\psi^2}{2\phi} - 2\kappa \hat{g}_{\zeta\zeta}} Q_0 g_{x\zeta}^2 = \frac{Q_0}{2Q_\zeta} g_{x\zeta} + \frac{Q_0}{2\phi r(r+Q_x)} - \gamma_M \left( \sigma_D^2 + \epsilon \sigma_\zeta^2 \frac{g_{x\zeta}^2}{4} \right).$$

If we take the limit, we get

$$-\frac{Q_0 Q_x}{r Q_\zeta} g_{x\zeta} - \frac{2\gamma_L \sigma_\zeta^2 \hat{g}_{\zeta\zeta}}{Q_\zeta g_{x\zeta} + \frac{\psi^2}{2\phi} - 2\kappa \hat{g}_{\zeta\zeta}} Q_0 g_{x\zeta}^2 = \frac{Q_0}{2Q_\zeta} g_{x\zeta} + \frac{Q_0}{2\phi r(r+Q_x)} - \gamma_M \sigma_D^2$$

Using the fact that  $\bar{X}_{ss} = Q_0/Q_x$  together with the limit in (A.15) we get

$$\bar{X}_{ss} = \frac{\gamma_M \sigma_D^2}{\frac{Q_x g_{x\zeta}}{2Q_\zeta} + \frac{Q_x}{2\phi r(r+Q_x)} + \frac{Q_x^2}{r Q_\zeta} g_{x\zeta} - \frac{Q_x g_{x\zeta}^2}{Q_\zeta g_{x\zeta} + \frac{\psi^2}{2\phi} + \frac{\kappa}{\gamma_L \sigma_\zeta^2}}$$

Hence, replacing  $\bar{\sigma}_\zeta^2 = \sigma_\zeta^2/2\kappa$  we get that

$$\bar{X}_{ss} = (1 + \Delta) \frac{\gamma_M}{\gamma_L + \gamma_M},$$

where

$$\Delta \equiv \frac{\gamma_L \psi^2 \bar{\sigma}_\zeta^2 \left( 1 + r\phi(\gamma_L + \gamma_M) \sigma_D^2 + \sqrt{1 + r\phi(\gamma_L + \gamma_M) \sigma_D^2 (9r\phi(\gamma_L + \gamma_M) \sigma_D^2 + 2)} \right)}{4\phi^2 (\gamma_L + \gamma_M)^2 \sigma_D^4 \left( \phi + \gamma_L \psi^2 \bar{\sigma}_\zeta^2 \right)},$$

which corresponds to the expression in the proposition after replacing  $R_q$  and doing some simplifications.

The final step is to find the coefficients  $P_x$  and  $P_\zeta$ . From equations (15c) and (A.16), we have that

$$P_\zeta = \frac{g_{x\zeta}}{2} = \frac{\psi}{4\phi} \frac{1}{Q_x} = \frac{\psi(\gamma_L + \gamma_M) \sigma_D^2}{4r\phi R_q}.$$

Next, from equations (15c), (A.16) and (A.21), we have that

$$\begin{aligned} P_x &= 2g_{xx} + \frac{Q_x}{Q_\zeta} g_{x\zeta} \\ &= 2g_{xx} + \frac{\alpha}{r\phi}. \end{aligned}$$

Moreover, taking the limit in equation (A.12) and replacing (A.16) and (A.21), we get that

$$g_{xx} = \frac{\alpha}{2r^2\phi}Q_x + \frac{1}{4r\phi} - \frac{\gamma_L}{2}\sigma_D^2.$$

Hence, we get

$$P_x = \frac{\alpha(r + Q_x)}{r^2\phi} + \frac{1}{2r\phi} - \gamma_L\sigma_D^2,$$

which reduces to the expressions in the statement of the proposition once we replace  $\alpha = r\phi(\gamma_L + \gamma_M)\sigma_D^2$ .

### Proof Proposition 5

Differentiating  $\bar{X}_{ss}$  in Proposition 4 we get

$$\begin{aligned} \frac{\partial \bar{X}_{ss}}{\partial \psi} &= \frac{\psi \bar{\sigma}_\zeta^2 \gamma_L \gamma_M \left( \sqrt{(1 + r\phi(\gamma_L + \gamma_M)\sigma_D^2)^2 + 8(\gamma_L + \gamma_M)^2 r^2 \phi^2 \sigma_D^4} - (1 + r\phi(\gamma_L + \gamma_M)\sigma_D^2) \right)}{2(\gamma_L + \gamma_M)^3 \phi \sigma_D^4 \left( \phi + \psi^2 \gamma_L \bar{\sigma}_\zeta^2 \right)^2} > 0 \\ \frac{\partial \bar{X}_{ss}}{\partial \bar{\sigma}_\zeta^2} &= \frac{\psi^2 \gamma_L \gamma_M \left( \sqrt{(1 + r\phi(\gamma_L + \gamma_M)\sigma_D^2)^2 + 8(\gamma_L + \gamma_M)^2 r^2 \phi^2 \sigma_D^4} - (1 + r\phi(\gamma_L + \gamma_M)\sigma_D^2) \right)}{2(\gamma_L + \gamma_M)^3 \phi \sigma_D^4 \left( \phi + \psi^2 \gamma_L \bar{\sigma}_\zeta^2 \right)^2} > 0 \end{aligned}$$

The sign of the derivative of  $\bar{X}_{ss}$  with respect to  $\sigma_D^2$  is given by

$$\text{sign} \frac{\partial}{\partial \sigma_D^2} \frac{rR_q}{\phi(\gamma_L + \gamma_M)^2 \sigma_D^4} = \text{sign} \frac{\partial}{\partial \alpha} \frac{\sqrt{(\alpha + 1)^2 + 8\alpha^2} - \alpha - 1}{\alpha^2} < 0.$$

The sign of the derivative of  $\bar{X}_{ss}$  with respect to  $\phi$  is given by

$$\text{sign} \frac{\partial}{\partial \phi} \frac{\gamma_L \psi^2 \bar{\sigma}_\zeta^2}{\phi + \gamma_L \psi^2 \bar{\sigma}_\zeta^2} \frac{rR_q}{\phi(\gamma_L + \gamma_M)^2 \sigma_D^4},$$

which is negative as both terms in the product are positive and decreasing in  $\phi$ . Using Proposition 1 we get that the mean steady state price is

$$\bar{p}_{ss} = \frac{\mu_D}{r} - \gamma_M \sigma_D^2 + \left( \frac{1}{2r\phi} + \gamma_M \sigma_D^2 \right) \bar{X}_{ss}.$$



The comparative statics for  $\psi, \phi, \bar{\sigma}_\zeta^2$  follow directly from the ones for  $\bar{X}_{ss}$ . The comparative statics for  $\sigma_D^2$  is direct as long as  $\bar{X}_{ss} < 1$  so the risk premium is positive.

## B Model with Private Benefits

### Proof Proposition 6

*Proof.* Using the definition of  $Z_t$  in (27), we get that

$$\begin{aligned}
dZ_t &= d\zeta_t + \frac{Q_b}{Q_\zeta} db_t \\
&= -\kappa\zeta_t dt - \lambda \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b \\
&= -\kappa \left( Z_t - \frac{Q_b}{Q_\zeta} b_t \right) dt - \lambda \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b \\
&= -\kappa Z_t dt + (\kappa - \lambda) \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b,
\end{aligned}$$

where in the third line we have use the relation

$$Z_t = \zeta_t + \frac{Q_b}{Q_\zeta} b_t.$$

On the other hand, given the conjectured equilibrium effort and the definition of  $Z_t$ , we can write the stochastic differential equations for the cumulative dividends process as

$$\begin{aligned}
dD_t &= (\mu_D + A_0 + A_x X_t + A_\zeta \zeta_t + A_b b_t + A_{\hat{b}} \hat{b}_t) dt + \sigma_D dB_t^D \\
&= \left( \mu_D + A_0 + A_x X_t + A_\zeta Z_t - A_\zeta \frac{Q_b}{Q_\zeta} b_t + A_b b_t + A_{\hat{b}} \hat{b}_t \right) dt + \sigma_D dB_t^D.
\end{aligned}$$

From here, we get a standard single dimensional filtering problem for  $b_t$  with the observation process

$$\begin{aligned}
dZ_t &= -\kappa Z_t dt + (\kappa - \lambda) \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b \\
dD_t &= \left( \mu_D + A_0 + A_x X_t + A_\zeta Z_t - A_\zeta \frac{Q_b}{Q_\zeta} b_t + A_b b_t + A_{\hat{b}} \hat{b}_t \right) dt + \sigma_D dB_t^D
\end{aligned}$$

Adapting the notation in [Liptser and Shiryaev \(2001a\)](#) to our problem we get

$$\begin{aligned}
a_0(t) &= 0 \\
a_1(t) &= -\lambda \\
b_1(t) &= \sigma_b \\
b_2(t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
A_0(t) &= \begin{pmatrix} -\kappa Z_t \\ \mu_D + A_0 + A_x X_t + A_b \hat{b}_t + A_\zeta Z_t \end{pmatrix} \\
A_1(t) &= \begin{pmatrix} (\kappa - \lambda) \frac{Q_b}{Q_\zeta} \\ A_b - A_\zeta \frac{Q_b}{Q_\zeta} \end{pmatrix} \\
B_1(t) &= \begin{pmatrix} \frac{Q_b}{Q_\zeta} \sigma_b \\ 0 \end{pmatrix} \\
B_2(t) &= \begin{pmatrix} \sigma_\zeta & 0 \\ 0 & \sigma_D \end{pmatrix}
\end{aligned}$$

Using Theorem 12.7 in [Lipster and Shiryaev](#) we get

$$\begin{aligned}
d\hat{b}_t &= -\lambda \hat{b}_t dt + \beta_q \left( dZ_t + \left( \kappa Z_t - (\kappa - \lambda) \frac{Q_b}{Q_\zeta} \hat{b}_t \right) dt \right) \\
&\quad + \beta_D \left( dD_t - \left( \mu_D + A_0 + A_x X_t + A_b \hat{b}_t + A_\zeta Z_t + \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \hat{b}_t \right) dt \right)
\end{aligned}$$

where

$$\begin{aligned}
\begin{pmatrix} \beta_q & \beta_D \end{pmatrix} &= (b_1 B_1^\top + b_2 B_2^\top + \sigma_b^2 A_1^\top) (B_1 B_1^\top + B_2 B_2^\top)^{-1} \\
0 &= 2a_1 \sigma_b^2 + b_1 b_1^\top + b_2 b_2^\top - (b_1 B_1^\top + b_2 B_2^\top + \sigma_b^2 A_1^\top) (B_1 B_1^\top + B_2 B_2^\top)^{-1} (b_1 B_1^\top + b_2 B_2^\top + \sigma_b^2 A_1^\top)^\top.
\end{aligned}$$

From here we get that  $(\beta_q, \beta_D)$  is given by

$$\begin{aligned}
\beta_q &= \frac{\sigma_b^2 + (\kappa - \lambda) \sigma_b^2 \frac{Q_b}{Q_\zeta}}{\left( \frac{Q_b}{Q_\zeta} \right)^2 \sigma_b^2 + \sigma_\zeta^2 Q_\zeta} \\
\beta_D &= \frac{\sigma_b^2}{\sigma_D^2} \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right)
\end{aligned}$$

and the stationary variance of  $\hat{b}$ ,  $\sigma_{\hat{b}}^2$ , is the positive root of the following quadratic equation.

$$0 = -2\lambda\sigma_{\hat{b}}^2 + \sigma_b^2 - \left[ \frac{(\sigma_b^2 + (\kappa - \lambda)\sigma_{\hat{b}}^2)^2}{\sigma_\zeta^2 + \left(\frac{Q_b}{Q_\zeta}\right)^2 \sigma_b^2} \left(\frac{Q_b}{Q_\zeta}\right)^2 - \frac{\sigma_{\hat{b}}^4}{\sigma_D^4} \left(A_b - A_\zeta \frac{Q_b}{Q_\zeta}\right)^2 \right]$$

Next, we express the stochastic differential equation for  $\hat{b}_t$  in term of the innovation processes  $\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D$ . We can write

$$\begin{aligned} dZ_t + \left( \kappa Z_t - (\kappa - \lambda) \frac{Q_b}{Q_\zeta} \hat{b}_t \right) dt &= \kappa \frac{Q_b}{Q_\zeta} (b_t - \hat{b}_t) dt + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \left( -\lambda b_t dt + \sigma_b dB_t^b + \lambda \hat{b}_t dt \right) \\ &= \kappa \frac{Q_b}{Q_\zeta} (b_t - \hat{b}_t) dt + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \\ &= \kappa (\hat{\zeta}_t - \zeta_t) dt + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \\ &= \sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \end{aligned}$$

where we have use the relation

$$Z_t = \zeta_t + \frac{Q_b}{Q_\zeta} b_t = \hat{\zeta}_t + \frac{Q_b}{Q_\zeta} \hat{b}_t.$$

Moreover, the previous relation also implies that  $\hat{\zeta}_t$  and  $\zeta_t$  are related as follows

$$\hat{\zeta}_t - \zeta_t = \frac{Q_b}{Q_\zeta} (b_t - \hat{b}_t).$$

Hence, we arrive to the SDE for  $\hat{b}_t$  in the proposition.

$$d\hat{b}_t = -\lambda \hat{b}_t dt + \beta_q \left( \sigma_\zeta d\tilde{B}_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b d\tilde{B}_t^b \right) + \beta_D \sigma_D d\tilde{B}_t^D$$

The final step is to find the SDE for  $\hat{\zeta}_t$ . By definition,  $d\hat{\zeta}_t = dZ_t - \frac{Q_b}{Q_\zeta} d\hat{b}_t$ , hence we can write

$$\begin{aligned} d\hat{\zeta}_t &= -\kappa Z_t dt + (\kappa - \lambda) \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b - \frac{Q_b}{Q_\zeta} d\hat{b}_t \\ &= -\kappa \left( \zeta_t + \frac{Q_b}{Q_\zeta} b_t \right) dt + (\kappa - \lambda) \frac{Q_b}{Q_\zeta} b_t + \sigma_\zeta dB_t^\zeta + \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b - \frac{Q_b}{Q_\zeta} d\hat{b}_t \\ &= -\kappa \hat{\zeta}_t dt + \sigma_\zeta \left( 1 - \frac{Q_b}{Q_\zeta} \beta_q \right) d\tilde{B}_t^\zeta + \sigma_b \frac{Q_b}{Q_\zeta} \left( 1 - \beta_q \frac{Q_b}{Q_\zeta} \right) d\tilde{B}_t^b - \sigma_D \beta_D \frac{Q_b}{Q_\zeta} d\tilde{B}_t^D, \end{aligned}$$

which corresponds to expression in the Proposition. Finally, by the innovation theorem (Liptser and Shiryaev, 2001b, Theorem 7.17), the processes  $\tilde{B}_t^\zeta, \tilde{B}_t^b, \tilde{B}_t^D$  are standard Brownian motions under  $\mathcal{F}_t^{q,D}$  ■

#### Proof Lemma 4

*Proof.* We derive the stochastic differential equation for  $\hat{b}_t$  given the blockholder filtration  $\mathcal{F}_t^{D,b,\zeta}$ . Given an arbitrary strategy  $\tilde{a}_t$  and  $\tilde{q}_t$  we have that

$$\begin{aligned} d\hat{b}_t &= -\lambda \hat{b}_t dt - \beta_D \left( A_0 + A_x X_t + A_{\hat{b}} \hat{b}_t + A_\zeta Z_t + \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \hat{b}_t - \tilde{a}_t \right) dt + \beta_D \sigma_D dB_t^D \\ &\quad + \beta_q \left( dZ_t + \left( \kappa Z_t - (\kappa - \lambda) \frac{Q_b}{Q_\zeta} \hat{b}_t \right) dt \right) \end{aligned}$$

Replacing  $dZ_t$  in equation (28) we get

$$\begin{aligned} d\hat{b}_t &= -\lambda \hat{b}_t dt - \beta_D \left( A_0 + A_x X_t + A_{\hat{b}} \hat{b}_t + A_\zeta Z_t + \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) \hat{b}_t - \tilde{a}_t \right) dt + \beta_q (\kappa - \lambda) \frac{Q_b}{Q_\zeta} (b_t - \hat{b}_t) dt \\ &\quad + \beta_q \sigma_\zeta dB_t^\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b + \beta_D \sigma_D dB_t^D. \end{aligned}$$

The result follows directly after replacing  $Z_t = \frac{\tilde{q}_t - Q_0 + Q_x X_t - \hat{b}_t}{Q_\zeta}$ , in which case we arrive to the following expression

$$\begin{aligned} d\hat{b}_t &= -\lambda \hat{b}_t dt - \beta_D \left( A_0 + \left( A_x + A_\zeta \frac{Q_x}{Q_\zeta} \right) X_t + A_\zeta \left( \frac{\tilde{q}_t - Q_0}{Q_\zeta} \right) + \left( A_\zeta \frac{-Q_b}{Q_\zeta} + A_b + A_{\hat{b}} \right) \hat{b}_t - \tilde{a}_t \right) dt \\ &\quad + \beta_q (\kappa - \lambda) \frac{Q_b}{Q_\zeta} (b_t - \hat{b}_t) dt + \beta_q \sigma_\zeta dB_t^\zeta + \beta_q \frac{Q_b}{Q_\zeta} \sigma_b dB_t^b + \beta_D \sigma_D dB_t^D, \end{aligned}$$

which, after defining the constants  $\mathcal{B}_0, \mathcal{B}_x, \mathcal{B}_b, \mathcal{B}_{\hat{b}}, \mathcal{B}_q$ , corresponds to the equation for  $\hat{b}_t$  in the statement of the lemma. ■

### Proof Lemma 7

*Proof.* The HJB equation for the competitive investor optimization problem is

$$\begin{aligned}
rJ = & \max_{c,q} u_M(c) + (rW - c - pq + (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}}) \hat{b}_t) Y) J_W - \kappa \hat{\zeta} J_{\hat{\zeta}} - \lambda \hat{b} J_{\hat{b}} \\
& + \left( Q_0 - Q_x X + Q_\zeta \hat{\zeta} + (Q_b + Q_{\hat{b}}) \hat{b} \right) J_X + q J_Y + \frac{1}{2} \left[ \sigma_D^2 Y^2 J_{WW} + \left( \beta_Z^2 \sigma_\zeta^2 + \sigma_b^2 \beta_Z^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 + \beta_D^2 \sigma_D^2 \right) J_{\hat{b}\hat{b}} \right. \\
& \quad \left. + \left( \sigma_\zeta^2 \left( 1 - \frac{Q_b}{Q_\zeta} \beta_Z \right)^2 + \sigma_b^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \left( 1 - \beta_Z \frac{Q_b}{Q_\zeta} \right)^2 + \sigma_D^2 \beta_D^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \right) J_{\hat{\zeta}\hat{\zeta}} \right. \\
& \quad \left. + 2\sigma_D^2 \beta_D Y J_{W\hat{b}} - 2\sigma_D^2 \beta_D Y \frac{Q_b}{Q_\zeta} J_{W\hat{\zeta}} + \left( \sigma_\zeta^2 \beta_Z \left( 1 - \frac{Q_b}{Q_\zeta} \beta_Z \right) + \sigma_b^2 \beta_Z \frac{Q_b}{Q_\zeta} \left( 1 - \beta_Z \frac{Q_b}{Q_\zeta} \right) - \sigma_D^2 \beta_D^2 \frac{Q_b}{Q_\zeta} \right) J_{\hat{b}\hat{\zeta}} \right]
\end{aligned}$$

As we did in the model without liquidity shocks, we conjecture a value function

$$J(W, Y, X, \hat{b}, \hat{\zeta}) = - \frac{\exp\left(-r\gamma_M \left(W_M + H(Y, X, \hat{b}, \hat{\zeta})\right)\right)}{r}$$

The first order condition for consumption is

$$u'_M(c) = J_W,$$

so

$$u_M(c) = rJ$$

and

$$c = rW_M + rH(Y, X, \hat{b}, \hat{\zeta})$$

Replacing our conjecture for the value function and the first order condition for consumption, and

defining Let's define

$$\begin{aligned}\Sigma_{\hat{b}} &= \beta_Z^2 \sigma_\zeta^2 + \sigma_b^2 \beta_Z^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 + \beta_D^2 \sigma_D^2 \\ \Sigma_{\hat{z}} &= \sigma_\zeta^2 \left( 1 - \frac{Q_b}{Q_\zeta} \beta_Z \right)^2 + \sigma_b^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \left( 1 - \beta_Z \frac{Q_b}{Q_\zeta} \right)^2 + \sigma_D^2 \beta_D^2 \left( \frac{Q_b}{Q_\zeta} \right)^2 \\ \Sigma_{\hat{b}\hat{z}} &= \sigma_\zeta^2 \beta_Z \left( 1 - \frac{Q_b}{Q_\zeta} \beta_Z \right) + \sigma_b^2 \beta_Z \frac{Q_b}{Q_\zeta} \left( 1 - \beta_Z \frac{Q_b}{Q_\zeta} \right) - \sigma_D^2 \beta_D^2 \frac{Q_b}{Q_\zeta}\end{aligned}$$

we get

$$\begin{aligned}rH &= \max_q (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta}_t + (A_b + A_{\hat{b}}) \hat{b}) Y - pq - \frac{r\gamma_M}{2} \left[ \sigma_D^2 Y^2 + 2\sigma_D^2 \beta_D \left( H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}} \right) Y \right. \\ &\quad \left. + \Sigma_{\hat{b}} H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}} \right] + \left( Q_0 - Q_x X + Q_\zeta \hat{\zeta} + (Q_b + Q_{\hat{b}}) \hat{b} \right) H_X \\ &\quad + qH_Y - \kappa \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2} \left[ \Sigma_{\hat{b}} H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}} \right].\end{aligned}$$

Using the first conditions for  $q$ ,  $p = H_Y$ , we get

$$\begin{aligned}rH &= (\mu_D + A_0 + A_x X + A_\zeta \hat{\zeta} + (A_b + A_{\hat{b}}) \hat{b}) Y - \frac{r\gamma_M}{2} \left[ \sigma_D^2 Y^2 + 2\sigma_D^2 \beta_D \left( H_{\hat{b}} - \frac{Q_b}{Q_\zeta} H_{\hat{\zeta}} \right) Y \right. \\ &\quad \left. + \Sigma_{\hat{b}} H_{\hat{b}}^2 + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}}^2 + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}} H_{\hat{\zeta}} \right] + \left( Q_0 - Q_x X + Q_\zeta \hat{\zeta} + (Q_b + Q_{\hat{b}}) \hat{b} \right) H_X \\ &\quad - \kappa \hat{\zeta} H_{\hat{\zeta}} - \lambda \hat{b} H_{\hat{b}} + \frac{1}{2} \left[ \Sigma_{\hat{b}} H_{\hat{b}\hat{b}} + \Sigma_{\hat{\zeta}} H_{\hat{\zeta}\hat{\zeta}} + 2\Sigma_{\hat{b}\hat{\zeta}} H_{\hat{b}\hat{\zeta}} \right].\end{aligned}$$

■

## Proof Lemma 8

*Proof.* The derivation follows the the proof in Lemma 7. The HJB equation is

$$\begin{aligned}rV &= \max_{c,a,q} u_L(c) + (rW - c - R(q, X, \hat{b})q - \Phi(a, \zeta) + (\mu_D + a + b)X)V_W - \kappa \zeta V_\zeta - \lambda b V_b \\ &\quad + \left( -\lambda \hat{b} + B_0 + B_x X + B_b b + B_{\hat{b}} \hat{b} + B_q q + \beta_D a \right) V_{\hat{b}} \\ &\quad + qV_X + \frac{1}{2} \left[ \sigma_D^2 X^2 V_{WW} + \Sigma_{\hat{b}} V_{\hat{b}\hat{b}} + \sigma_b^2 V_{bb} + \sigma_\zeta V_{\zeta\zeta} + 2\sigma_D^2 \beta_D X V_{W\hat{b}} + \beta_Z \sigma_\zeta^2 V_{\hat{b}\zeta} + \beta_Z \frac{Q_b}{Q_\zeta} \sigma_b^2 V_{\hat{b}b} \right]\end{aligned}$$

We conjecture a value function

$$V(W, X, \zeta, b, \hat{b}) = -\frac{\exp\left(-r\gamma_L\left(W + G(X, \zeta, b, \hat{b})\right)\right)}{r}.$$

Replacing the value function and the first order condition for consumption, we get

$$\begin{aligned} rG = \max_{a,q} & (\mu_D + a + b)X - R(q, X, \hat{b})q - \Phi(a, \zeta) \\ & - \frac{r\gamma_L}{2} \left[ \sigma_D^2 X^2 + 2\sigma_D^2 \beta_D G_{\hat{b}} X + \Sigma_{\hat{b}} G_{\hat{b}}^2 + \sigma_b^2 G_b^2 + \sigma_\zeta G_\zeta^2 + \beta_Z \sigma_\zeta^2 G_{\hat{b}} G_\zeta + \beta_Z \frac{Q_b}{Q_\zeta} \sigma_b^2 G_{\hat{b}} G_b \right] \\ & - \kappa \zeta G_\zeta - \lambda b G_b + \left( -\lambda \hat{b} + \mathcal{B}_0 + \mathcal{B}_x X + \mathcal{B}_b b + \mathcal{B}_{\hat{b}} \hat{b} + \mathcal{B}_q q + \beta_D a \right) G_{\hat{b}} + q G_X \\ & + \frac{1}{2} \left[ \Sigma_{\hat{b}} G_{\hat{b}\hat{b}} + \sigma_b^2 G_{bb} + \sigma_\zeta G_{\zeta\zeta} + \beta_Z \sigma_\zeta^2 G_{\hat{b}\zeta} + \beta_Z \frac{Q_b}{Q_\zeta} \sigma_b^2 G_{\hat{b}b} \right]. \end{aligned}$$

■

## B.1 System of Equations Equilibrium

The first step in the determination of the equilibrium is to determine the coefficients of the certainty equivalent. The system of equations determining the coefficients for the quadratic terms is decoupled from the system of equations determining the linear terms. After solving for the quadratic terms, we can determine the rest of the coefficients by solving a system of linear equations. Replacing the conjecture certainty equivalent  $H$  in Lemma 7 we find that

$$h_{xy} = \frac{rA_x}{r + Q_x} \tag{B.1a}$$

$$h_{y\hat{\zeta}} = \frac{A_\zeta + Q_\zeta h_{xy}}{r + \kappa} \tag{B.1b}$$

$$h_{y\hat{b}} = \frac{A_b + A_{\hat{b}} + h_{xy}(Q_b + Q_{\hat{b}})}{r + \lambda} \tag{B.1c}$$

$$h_{yy} = -\frac{\gamma_M}{2} \left( \sigma_D^2 + \Sigma_{\hat{b}}^2 h_{y\hat{b}}^2 + \Sigma_{\hat{\zeta}}^2 h_{y\hat{\zeta}}^2 + 2\beta_D \sigma_D^2 \left( h_{y\hat{b}} - \frac{Q_b}{Q_\zeta} h_{y\hat{\zeta}} \right) + 2\Sigma_{\hat{b}\hat{\zeta}}^2 h_{y\hat{b}} h_{y\hat{\zeta}} \right) \tag{B.1d}$$

Replacing the conjectured certainty equivalent for the blockholder in the first order conditions (36a) and (36b), we get that the coefficients  $\mathbf{A}, \mathbf{Q}$  are given by

$$Q_0 = \frac{g_x - R_0 + \mathcal{B}_q g_{\hat{b}}}{2R_q} \quad (\text{B.2a})$$

$$Q_x = \frac{R_x - 2g_{xx} - \mathcal{B}_q g_{x\hat{b}}}{2R_q} \quad (\text{B.2b})$$

$$Q_\zeta = \frac{g_{x\zeta} + \mathcal{B}_q g_{\zeta\hat{b}}}{2R_q} \quad (\text{B.2c})$$

$$Q_b = \frac{g_{xb} + \mathcal{B}_q g_{b\hat{b}}}{2R_q} \quad (\text{B.2d})$$

$$Q_{\hat{b}} = \frac{g_{x\hat{b}} - R_{\hat{b}} + 2\mathcal{B}_q g_{\hat{b}\hat{b}}}{2R_q} \quad (\text{B.2e})$$

$$A_0 = \beta_D \frac{g_{\hat{b}}}{2\phi} \quad (\text{B.2f})$$

$$A_x = \frac{1 + \beta_D g_{x\hat{b}}}{2\phi} \quad (\text{B.2g})$$

$$A_\zeta = \frac{\psi + \beta_D g_{\zeta\hat{b}}}{2\phi} \quad (\text{B.2h})$$

$$A_b = \beta_D \frac{g_{b\hat{b}}}{2\phi} \quad (\text{B.2i})$$

$$A_{\hat{b}} = \beta_D \frac{g_{\hat{b}\hat{b}}}{2\phi} \quad (\text{B.2j})$$

Also, replacing the conjectured certainty equivalent in the HJB equation we get that the coefficients for the higher order solve

$$r g_{xx} = A_x - \phi A_x^2 - 2Q_x g_{xxx} + g_{x\hat{b}} (\mathcal{B}_x - \mathcal{B}_q Q_x) + Q_x (R_x - R_q Q_x) \quad (\text{B.3a})$$

$$- \frac{r\gamma L}{2} \left( \Sigma_{\hat{b}}^2 g_{x\hat{b}}^2 + \sigma_D^2 + \sigma_b^2 g_{xb}^2 + \sigma_\zeta^2 g_{x\zeta}^2 + 2\beta_D \sigma_D g_{x\hat{b}}^2 + 2\beta_q \sigma_\zeta^2 g_{x\hat{b}} g_{x\zeta} + 2\beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} g_{xb} g_{x\hat{b}} \right)$$

$$(r + \kappa) g_{x\zeta} = A_\zeta + \psi A_x - Q_\zeta (R_x - 2g_{xx}) + \mathcal{B}_x g_{\zeta\hat{b}} - Q_x g_{x\zeta} + 2Q_x Q_\zeta R_q + \mathcal{B}_q Q_\zeta g_{x\hat{b}} \quad (\text{B.3b})$$

$$- 2\phi A_x A_\zeta - \mathcal{B}_q Q_x g_{\zeta\hat{b}} - r\gamma L \left( \beta_D \sigma_D^2 g_{\zeta\hat{b}} + \sigma_b^2 g_{xb} g_{\zeta b} + 2\sigma_\zeta^2 g_{x\zeta} g_{\zeta\zeta} + \Sigma_{\hat{b}}^2 g_{x\hat{b}} g_{\zeta\hat{b}} + \frac{Q_b}{Q_\zeta} \beta_q \sigma_b^2 (g_{xb} g_{\zeta\hat{b}} + g_{x\hat{b}} g_{\zeta b}) + \beta_q \sigma_\zeta^2 (2g_{x\hat{b}} g_{\zeta\zeta} + g_{x\zeta} g_{\zeta\hat{b}}) \right)$$

$$(r + \lambda) g_{xb} = \delta + A_b + 2Q_b g_{xxx} - Q_x g_{xb} + g_{x\hat{b}} (\mathcal{B}_b + \mathcal{B}_q Q_b) + g_{b\hat{b}} (\mathcal{B}_x - \mathcal{B}_q Q_x) \quad (\text{B.3c})$$

$$- Q_b (R_x - Q_x R_q) - 2\phi A_b A_x + Q_b Q_x R_q - r\gamma L \left( 2\sigma_b^2 g_{bb} g_{xb} + \sigma_\zeta^2 g_{x\zeta} g_{\zeta b} + \Sigma_{\hat{b}}^2 g_{b\hat{b}} g_{x\hat{b}} + \beta_D \sigma_D^2 g_{b\hat{b}} + \beta_q \sigma_\zeta^2 (g_{b\hat{b}} g_{x\zeta} + g_{x\hat{b}} g_{\zeta b}) + 2\beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} (g_{bb} g_{x\hat{b}} + g_{b\hat{b}} g_{xb}) \right)$$



$$\begin{aligned}
(r + \lambda)g_{x\hat{b}} &= A_{\hat{b}} - 2\phi A_{\hat{b}}A_x + 2Q_{\hat{b}}g_{xx} - Q_x g_{x\hat{b}} + g_{x\hat{b}}(\mathcal{B}_{\hat{b}} + \mathcal{B}_q Q_{\hat{b}}) + 2g_{\hat{b}\hat{b}}(\mathcal{B}_x - \mathcal{B}_q Q_x) \\
&\quad + Q_x(R_{\hat{b}} + Q_{\hat{b}}R_q) - Q_{\hat{b}}(R_x - Q_x R_q) - r\gamma_L \left( \sigma_b^2 g_{bb} g_{xb} + \sigma_\zeta^2 g_{x\zeta} g_{\zeta\hat{b}} \right. \\
&\quad \left. + 2\Sigma_{\hat{b}\hat{b}}^2 g_{\hat{b}\hat{b}} g_{x\hat{b}} + 2\beta_D \sigma_D^2 g_{\hat{b}\hat{b}} + \beta_q \sigma_\zeta^2 (2g_{\hat{b}\hat{b}} g_{x\zeta} + g_{x\hat{b}} g_{\zeta\hat{b}}) + \beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} (g_{\hat{b}\hat{b}} g_{x\hat{b}} + 2g_{\hat{b}\hat{b}} g_{xb}) \right)
\end{aligned} \tag{B.3d}$$

$$\begin{aligned}
(r + 2\kappa)g_{\zeta\zeta} &= \psi A_\zeta + Q_\zeta g_{x\zeta} - Q_\zeta^2 R_q - A_\zeta^2 \phi + \mathcal{B}_q Q_\zeta g_{\zeta\hat{b}} \\
&\quad - \frac{r\gamma_L}{2} \left( \Sigma_{\hat{b}\hat{b}}^2 g_{\zeta\hat{b}}^2 + \sigma_b^2 g_{\zeta\hat{b}}^2 + 4\sigma_\zeta^2 g_{\zeta\zeta}^2 + 4\beta_q \sigma_\zeta^2 g_{\zeta\zeta} g_{\zeta\hat{b}} + 2\beta_q \sigma_b^2 \frac{Q_b g_{\zeta\hat{b}}}{Q_\zeta} g_{\zeta\hat{b}} \right)
\end{aligned} \tag{B.3e}$$

$$\begin{aligned}
(r + \kappa + \lambda)g_{\zeta b} &= \psi A_b + Q_b g_{x\zeta} + Q_\zeta g_{xb} + g_{\zeta\hat{b}}(\mathcal{B}_b + \mathcal{B}_q Q_b) - 2A_b A_z \phi - 2Q_b Q_\zeta R_q + \mathcal{B}_q Q_\zeta g_{b\hat{b}} \\
&\quad - r\gamma_L \left( 2\sigma_b^2 g_{bb} g_{\zeta b} + 2\sigma_\zeta^2 g_{\zeta b} g_{\zeta\zeta} + \Sigma_{\hat{b}\hat{b}}^2 g_{b\hat{b}} g_{\zeta\hat{b}} + \beta_q \sigma_\zeta^2 (2g_{b\hat{b}} g_{\zeta\zeta} + g_{\zeta b} g_{\zeta\hat{b}}) \right. \\
&\quad \left. + \beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} (2g_{bb} g_{\zeta\hat{b}} + g_{b\hat{b}} g_{\zeta b}) \right)
\end{aligned} \tag{B.3f}$$

$$\begin{aligned}
(r + \kappa + \lambda)g_{\zeta\hat{b}} &= \psi A_{\hat{b}} - 2\phi A_{\hat{b}}A_z + g_{x\zeta} + Q_\zeta g_{x\hat{b}} + g_{\zeta\hat{b}}(\mathcal{B}_{\hat{b}} + \mathcal{B}_q) - Q_\zeta(R_{\hat{b}} + R_q) \\
&\quad - Q_\zeta R_q + 2\mathcal{B}_q Q_\zeta g_{b\hat{b}} - r\gamma_L \left( \sigma_b^2 g_{b\hat{b}} g_{\zeta b} + 2\sigma_\zeta g_{\zeta\zeta} g_{\zeta\hat{b}}^2 + \beta_q \sigma_\zeta^2 g_{\zeta\hat{b}}^2 + 2\Sigma_{\hat{b}\hat{b}}^2 g_{\hat{b}\hat{b}} g_{\zeta\hat{b}} \right. \\
&\quad \left. + 4\beta_q \sigma_\zeta^2 g_{\hat{b}\hat{b}} g_{\zeta\zeta} + \beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} (g_{\hat{b}\hat{b}} g_{\zeta\hat{b}} + 2g_{\hat{b}\hat{b}} g_{\zeta b}) \right)
\end{aligned} \tag{B.3g}$$

$$\begin{aligned}
(r + 2\lambda)g_{bb} &= Q_b g_{xb} + g_{b\hat{b}}(\mathcal{B}_b + \mathcal{B}_q Q_b) - Q_b^2 R_q - A_b^2 \phi \\
&\quad - \frac{r\gamma_L}{2} \left( \Sigma_{\hat{b}\hat{b}}^2 g_{bb}^2 + 4\sigma_b^2 g_{bb}^2 + \sigma_\zeta^2 g_{\zeta b}^2 + 2\beta_q \sigma_\zeta^2 g_{b\hat{b}} g_{\zeta b} + 4\beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} g_{bb} g_{b\hat{b}} \right)
\end{aligned} \tag{B.3h}$$

$$\begin{aligned}
(r + 2\lambda)g_{\hat{b}\hat{b}} &= Q_b g_{x\hat{b}} + g_{xb} + g_{b\hat{b}}(\mathcal{B}_{\hat{b}} + \mathcal{B}_q) + 2g_{\hat{b}\hat{b}}(\mathcal{B}_b + \mathcal{B}_q Q_b) - Q_b(R_{\hat{b}} + R_q) \\
&\quad - 2\phi A_b A_{\hat{b}} - Q_b R_q - r\gamma_L \left( 2\sigma_b^2 g_{bb} g_{b\hat{b}} + \sigma_\zeta^2 g_{\zeta b} g_{\zeta\hat{b}} + 2\Sigma_{\hat{b}\hat{b}}^2 g_{b\hat{b}} g_{\hat{b}\hat{b}} \right. \\
&\quad \left. + \beta_q \sigma_\zeta^2 (g_{b\hat{b}} g_{\zeta\hat{b}} + 2g_{\hat{b}\hat{b}} g_{\zeta b}) + \beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} (g_{b\hat{b}}^2 + 4g_{bb} g_{b\hat{b}}) \right)
\end{aligned} \tag{B.3i}$$

$$\begin{aligned}
(r + 2\lambda)g_{\hat{b}\hat{b}} &= -\phi A_{\hat{b}}^2 + g_{x\hat{b}} + 2g_{\hat{b}\hat{b}}(\mathcal{B}_{\hat{b}} + \mathcal{B}_q) - (R_{\hat{b}} + R_q) \\
&\quad - \frac{r\gamma_L}{2} \left( 4\Sigma_{\hat{b}\hat{b}}^2 g_{\hat{b}\hat{b}}^2 + \sigma_b^2 g_{\hat{b}\hat{b}}^2 + \sigma_\zeta^2 g_{\zeta\hat{b}}^2 + 4\beta_q \sigma_\zeta^2 g_{\hat{b}\hat{b}} g_{\zeta\hat{b}} + 4\beta_q \sigma_b^2 \frac{Q_b}{Q_\zeta} g_{b\hat{b}} g_{\hat{b}\hat{b}} \right)
\end{aligned} \tag{B.3j}$$

Finally, using equations (32) and (33) we can write the coefficients  $\mathbf{R}$  in term of the coefficients  $\mathbf{h}$ . Thus, an equilibrium is given by a solution to system of 15 equation given by (B.1), (B.3) together with (31). The solution must satisfy the stationarity condition  $Q_x > 0$  and the second order condition  $R_q > 0$ .

## B.2 Impulse Response Functions

In order to compute the impulse response function we use the following results that can be found in [Evans \(2012\)](#).

**Lemma 5.** *The solution to the linear SDE*

$$d\mathbf{X}_t = (\mathbf{c} + \mathbf{D}\mathbf{X}_t)dt + \mathbf{E}d\mathbf{W}_t.$$

is

$$\mathbf{X}_t = e^{\mathbf{D}t}\mathbf{X}_0 + \int_0^t e^{\mathbf{D}(t-s)}(\mathbf{c}ds + \mathbf{E}d\mathbf{W}_s),$$

where  $e^{\mathbf{D}t}$  is the matrix exponential.

Next we derive the impulse response function. We start deriving the impulse response functions under  $\mathcal{F}_t^M$ . The blockholder block size is determined by the solution to the following linear system of stochastic differential equation

$$\begin{aligned} \begin{pmatrix} dX_t \\ d\hat{\zeta}_t \\ d\hat{b}_t \end{pmatrix} &= \left[ \begin{pmatrix} Q_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -Q_x & Q_\zeta & Q_b + Q_{\hat{b}} \\ 0 & -\kappa & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} X_t \\ \hat{\zeta}_t \\ \hat{b}_t \end{pmatrix} \right] dt \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) & \sigma_b \frac{Q_b}{Q_\zeta} \left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) & -\sigma_D \beta_D \frac{Q_b}{Q_\zeta} \\ \sigma_\zeta \beta_q & \sigma_b \beta_q \frac{Q_b}{Q_\zeta} & \sigma_D \beta_D \end{pmatrix} \begin{pmatrix} d\tilde{B}_t^\zeta \\ d\tilde{B}_t^b \\ d\tilde{B}_t^D \end{pmatrix} \end{aligned}$$

The solution to this equation is (see, e.g. [Evans \(2012\)](#))

$$\begin{pmatrix} X_t \\ \hat{\zeta}_t \\ \hat{b}_t \end{pmatrix} = \Pi(t) \begin{pmatrix} X_0 \\ \hat{\zeta}_0 \\ \hat{b}_0 \end{pmatrix} + \int_0^t \Pi(t-s) \left[ \begin{pmatrix} Q_0 \\ 0 \\ 0 \end{pmatrix} ds + \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta \left(1 - \frac{Q_b}{Q_\zeta}\beta_q\right) & \sigma_b \frac{Q_b}{Q_\zeta} \left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) & -\sigma_D \beta_D \frac{Q_b}{Q_\zeta} \\ \sigma_\zeta \beta_q & \sigma_b \beta_q \frac{Q_b}{Q_\zeta} & \beta_D \sigma_D \end{pmatrix} \begin{pmatrix} d\tilde{B}_s^\zeta \\ d\tilde{B}_s^b \\ d\tilde{B}_s^D \end{pmatrix} \right],$$

where

$$\Pi(t) = \begin{pmatrix} e^{-Q_x t} & \frac{(e^{-\kappa t} - e^{-Q_x t})Q_\zeta}{Q_x - \kappa} & \frac{(e^{-\lambda t} - e^{-Q_x t})(Q_b + Q_{\hat{b}})}{Q_x - \lambda} \\ 0 & e^{-\kappa t} & 0 \\ 0 & 0 & e^{-\lambda t} \end{pmatrix}.$$

Next, we calculate the matrix of impulse responses under  $\mathcal{F}_t^L$ , which we denote by  $\hat{\mathbf{I}}\mathbf{R}(t)$  and we let  $\hat{I}\mathbf{R}_j^i(t)$  be the impulse response function of  $j \in \{X_t, \hat{\zeta}_t, \hat{b}_t\}$  for an unexpected shock to

$i \in \{\hat{\zeta}_0, \hat{b}_0, D_0\}$ . The matrix  $\hat{\mathbf{I}}\mathbf{R}(t)$  is given by

$$\hat{\mathbf{I}}\mathbf{R}(t) = \Pi(t) \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta \left(1 - \frac{Q_b}{Q_\zeta} \beta_q\right) & \sigma_b \frac{Q_b}{Q_\zeta} \left(1 - \beta_q \frac{Q_b}{Q_\zeta}\right) & -\sigma_D \beta_D \frac{Q_b}{Q_\zeta} \\ \sigma_\zeta \beta_q & \sigma_b \beta_q \frac{Q_b}{Q_\zeta} & \sigma_D \beta_D \end{pmatrix}.$$

The impulse response functions for the block size is  $\hat{\mathbf{I}}\mathbf{R}(t)^\top \mathbf{e}_1$ , where  $\mathbf{e}_1$  is a  $3 \times 1$  vector with a one in the first entry and zero otherwise. From here, we can also compute the impulse response of effort which is given by

$$\begin{pmatrix} \hat{I}R_a^\zeta(t) \\ \hat{I}R_a^b(t) \\ \hat{I}R_a^D(t) \end{pmatrix} = \mathbf{I}\mathbf{R}(t)^\top \begin{pmatrix} A_x \\ A_\zeta \\ A_b + A_b \end{pmatrix}$$

Similarly, we derive the impulse response functions given the blockholder information set. We consider the following linear system of stochastic differential equation

$$\begin{pmatrix} dX_t \\ d\zeta_t \\ db_t \\ d\hat{b}_t \end{pmatrix} = \left[ \begin{pmatrix} Q_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -Q_x & Q_\zeta & Q_b & Q_b \\ 0 & -\kappa & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & \eta & -(\lambda + \eta) \end{pmatrix} \begin{pmatrix} X_t \\ \zeta_t \\ b_t \\ \hat{b}_t \end{pmatrix} \right] dt + \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta & 0 & 0 \\ 0 & \sigma_b & 0 \\ \beta_q \sigma_\zeta & \beta_q \frac{Q_b}{Q_\zeta} \sigma_b & \beta_D \sigma_D \end{pmatrix} \begin{pmatrix} dB_t^\zeta \\ dB_t^b \\ dB_t^D \end{pmatrix}$$

where

$$\eta \equiv \beta_D \left( A_b - A_\zeta \frac{Q_b}{Q_\zeta} \right) + \beta_q (\kappa - \lambda) \frac{Q_b}{Q_\zeta}$$

If we let

$$M \equiv \begin{pmatrix} -Q_x & Q_\zeta & Q_b & Q_b \\ 0 & -\kappa & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & \eta & -(\lambda + \eta) \end{pmatrix},$$

and denote the impulse response given the blockholder information set by  $\mathbf{I}\mathbf{R}(t)$ , then we get that the impulse response function is

$$\mathbf{I}\mathbf{R}(t) = e^{Mt} \begin{pmatrix} 0 & 0 & 0 \\ \sigma_\zeta & 0 & 0 \\ 0 & \sigma_b & 0 \\ \beta_q \sigma_\zeta & \beta_q \frac{Q_b}{Q_\zeta} \sigma_b & \beta_D \sigma_D \end{pmatrix}.$$

From here, we can immediately get the impulse response of effort, which is given by

$$IR_a(t) = \mathbf{IR}(t) \begin{pmatrix} A_x \\ A_\zeta \\ A_b \\ A_{\hat{b}} \end{pmatrix}.$$

For the impulse response for the price, we notice that

$$\hat{\zeta}_t = \zeta_t + \frac{Q_b}{Q_\zeta}(b_t - \hat{b}_t),$$

which means that

$$IR_p(t) = \mathbf{IR}(t) \begin{pmatrix} P_x \\ P_\zeta \\ P_\zeta \frac{Q_b}{Q_\zeta} \\ P_b - P_\zeta \frac{Q_b}{Q_\zeta} \end{pmatrix}.$$