PRIVATE CONTRACTS IN TWO-SIDED MARKETS

GASTÓN LLANES AND FRANCISCO RUIZ-ALISEDA

Abstract. We study a two-sided market in which a platform mediates between sellers and buyers, and signs private contracts with sellers. We compare this market with (i) a one-sided market with public contracts, and (ii) a two-sided market with public contracts. We find that equilibrium royalties can be positive or negative in a one-sided market with public contracts, are negative in a two-sided market with public contracts, and are positive in a two-sided market with private contracts. Thus, the privacy of contracts has a significant effect on the equilibrium price structure. We also show that private contracts lead to lower profit, consumer surplus, and welfare than public contracts.

Keywords: Two-sided markets, platforms, vertical relations, private contracts.

1. Introduction

Private contracts are common in many two-sided markets. For example, Amazon signs private contracts with publishers, Netflix with movie studios, Sony with game developers, HMOs with health-care providers, Google with phone manufacturers, and Apple with cellphone carriers. Our objective in this paper is to show that private contracting has a critical impact on price structures, platform profitability and welfare, and that it can help explain many commonly observed features of two-sided markets.

We examine these issues through a two-stage model in which a platform mediates between sellers and consumers. Sellers’ products may be substitutes or complements. In the first stage, the platform provider chooses how much to charge consumers and sellers to get access to the platform, as well as how high the (per-unit) royalty fees to be paid by sellers should be; next, sellers decide whether to accept the two-part-tariff contract offered by the platform provider, and consumers then decide whether or not to buy access to the platform. In the final stage, sellers post prices, and consumers who have purchased access to the platform choose how much to buy from sellers.

Our aim is to compare different information structures to shed light on how each of them affects royalty fees and access prices, as well as the platform provider’s
profits and welfare. In particular, we compare situations where the platform provider’s pricing scheme is publicly observable -the standard assumption in the two-sided market literature- with cases in which the platform’s offer to each seller is observed only by this seller. We refer to the latter situations as the case of “private contracts”, noting that they correspond to what one observes in many industries.

When the contract offered to a seller is private, the formation of beliefs by sellers and consumers when observing out-of-equilibrium play is critical. In line with past literature, we assume that consumers form “passive beliefs” (Hart and Tirole, 1990; Hagiu and Halaburda, 2014) when they observe a price for the platform that differs from the one they were expecting in equilibrium. In turn, a seller that observes an unexpected two-part tariff forms “wary beliefs” about the offer made to the other seller (McAfee and Schwartz, 1994; Rey and Vergé, 2004): it believes that the platform provider is acting opportunistically when pursuing such a deviation from expected play.¹

The contribution of our paper is to show that the conclusions drawn from a model of two-sided market with public contracts and one with private contracts stand in stark contrast. On the one hand, when contracts are public, we find that royalty fees are negative and the platform provider’s markup on consumers is positive. When contracts are private, on the other hand, royalty fees are positive and the platform provider’s markup on consumers is negative. These results fit well with the price patterns observed in many industries in which contracts are private (such as videogames and ebooks), and they do not depend on whether sellers’ products are substitutes or complements. We also find that, relative to the case of public contracts, private contracting results in lower profit for the platform provider as well as lower consumer and social surplus. Again, this finding does not depend on the nature of the goods offered by sellers (i.e., whether price competition exhibits strategic complementarity or substitutability).

There are several changes when contract offers that had a public character become private to the seller receiving it. To get a better sense of what drives our findings, it is therefore useful to study an intermediate case in which sellers observe all contract offers, but contracts are still unobserved by consumers. In this intermediate case in which consumers do not observe deviations from expected contracts, they fear being taken advantage of by the platform, so they (correctly)

¹Wary beliefs mitigate the opportunistic behavior of the platform provider relative to having sellers form passive beliefs. It is well-known from Rey and Vergé (2004) that there might exist no equilibria if sellers form passive beliefs in a setting like ours.
anticipate that royalties will be set to induce collusive pricing by sellers. Consumers can thus be induced to consume the platform only by having a negative markup on them. Compared to the public contracting case in which consumers observe the royalties charged to sellers, consumers rightly anticipate an incentive to raise the royalty fees and to sell the platform at a lower price. The profitability of the platform provider suffers when consumers cannot observe the offers it makes to sellers.

Consider next what happens if, in addition to assuming that consumers do not observe sellers’ contracts, we assume that each seller observes only the contract it is offered. One may be tempted to extrapolate Rey and Vergé’s (2004) finding when goods are substitutes that the platform provider must be worse off (relative to the intermediate case mentioned above), for it loses part of its market power vis-à-vis sellers that fear being taken advantage of given that it deals with each on a one-on-one basis. Such an extrapolation would be incorrect because it would miss the feedback loops to which a two-sided market gives rise.

In a two-sided market framework, decreasing the market power on one side may enhance market power on the other side: in particular, the loss in the platform provider’s market power with regards to sellers acts as a commitment to create a larger value to consumers. Relative to the cases in which it is only consumers who cannot observe offers made to sellers, the platform earns less through sellers keeping the number of consumers fixed (as shown by Rey and Vergé, 2004), but the lower prices charged by sellers make the platform more appealing to consumers. This allows the platform provider to charge a higher price to consumers and still attract more. The platform provider increases usage of the platform by consumers and increases its price, effects that dominate the lower profit extracted from sellers for each consumer attracted to the platform.

When sellers do not offer substitutes but rather offer complements, the platform earns less from sellers keeping the number of consumers fixed and also attracts fewer consumers (relative to when it is only consumers who cannot observe offers made to sellers). In this case, the loss of market power by the platform provider makes it less capable of internalizing the double marginalization problem faced by the sellers when pricing (Cournot, 1838), so consumers expect sellers to charge higher prices, and the platform becomes less valuable to them. Even though the platform provider charges lower prices to attract consumers, platform sales

---

2To the best of our knowledge, this setting has not been analyzed by the vertical relations literature dealing with secret contracts.
decrease and the platform provider is harmed by the lower usage of the platform by consumers and the smaller profit appropriated from sellers.

Based on these insights, it holds when contracts are private rather than public that consumers fear being taken advantage of by the platform because they cannot observe the actual royalties that the platform will receive from sellers. When sellers offer substitutes, this concern is mitigated because of the loss in market power that the platform provider bears when it secretly contracts with each seller, but our contribution in this case is to show that the consumers’ initial concern is not mitigated enough by this loss in control. As a result, the platform provider can only attract consumers by bearing a negative markup on them (given that they anticipate high royalties and hence high prices charged by sellers). The platform provider’s profits are therefore smaller when contracts are private rather than public. Both consumer surplus and social welfare decrease as well.

When sellers offer complements instead of substitutes, our contribution is to show that the consumers’ concern about the platform provider’s opportunistic behavior is accentuated because it has less control over the double marginalization problem faced by the sellers. Relative to public contracting, private contracts again result in higher royalties, higher prices charged by sellers, lower prices for the platform and lower profitability for the platform provider. Similarly for consumer and social welfare.

2. The model

We consider a model with \(n+1\) firms and a continuum of consumers. Firm 0 is a platform provider and produces a platform good (such as a video console) at a normalized marginal cost of zero. Firms \(i = 1, \ldots, n\) are sellers of platform-specific products (such as video games). These products can only be used by consumers who buy the platform. Sellers produce at zero marginal cost (again a normalization).

Consumers are uniformly spread on the positive real line and firm 0 is located at the left end. Given a consumer at distance \(x \in [0, \infty)\) from firm 0, consider her utility if she purchases one unit of the product sold by the firm at price \(p_0\) and purchases \(q_i \geq 0\) units of the product sold by firm \(i \in \{1, \ldots, n\}\) at price of \(p_i\) per unit. We assume (see Vives, 2001, for example) that such utility equals

\[
U_x(p_0, p_1, q_1, \ldots, p_n, q_n) = u(p_1, q_1, \ldots, p_n, q_n) - x - p_0,
\]
where
\[ u(p_1, q_1, \ldots, p_n, q_n) = \sum_{i=1}^{n} q_i - \frac{1}{2} \left( \sum_{i=1}^{n} q_i^2 + \theta \sum_{i=1}^{n} \sum_{j=1; j \neq i}^{n} q_i q_j \right) - \sum_{i=1}^{n} p_i q_i. \]

Parameter \( \theta \in (-1, 1) \) captures the degree of complementarity/substitution between sellers’ products. If \( \theta < 0 \), goods are complements, with their degree of complementarity decreasing with \( \theta \). If \( \theta = 0 \), goods are independent, whereas \( \theta > 0 \) implies that goods are substitutes, with their degree of substitutability increasing as \( \theta \) grows.

We consider the following two-stage model. In the first stage, the platform provider offers contracts to sellers and sets a price/markup \( p_0 \) for consumers. Sellers decide whether to accept the contract, and then consumers observe both \( p_0 \) and how many sellers have accepted the contract before having to decide whether to buy the platform good. In the second stage, sellers set prices for their products, and consumers decide how many products to buy.\(^3\)

A contract between seller \( i \in \{1, \ldots, n\} \) and the platform provider consists of a fixed fee \( f_i \) and a per-unit royalty fee \( w_i \).\(^4\) If seller \( i \) accepts the contract and then sells \( Q_i \) units to consumers, its total payment to the platform provider is \( f_i + w_i Q_i \).

In the first part of the paper, we assume \( n = 2 \) and study several games. In Section 3, we study a one-sided market with public contracts. That is, we assume that \( p_0 = 0 \), and that consumers and sellers observe all contracts before making their decisions. In Section 4, we study a two-sided market with public contracts. This game is analogous to the previous one, except that we allow for \( p_0 \neq 0 \). Finally, in Section 5, we study a two-sided market with private contracts. We first examine a situation in which it is only consumers who do not observe any of the contracts offered to sellers. We then examine a situation in which consumers do not observe any of the contracts offered to sellers, and each seller only observes the contract it is offered by the platform provider. We assume throughout that \( p_0 \) is contractible and it is written in the contract offered to any seller.\(^5\)

\(^3\)Our timing reflects the fact that consumers use the platform for many periods, during which platform-specific products are continuously being launched. For instance, buyers of a video console often buy it without observing the prices charged for the games they will consume during the lifetime of the console.

\(^4\)Our main results do not depend on fixed fees being available. The proof is available on request.

\(^5\)Even if \( p_0 \) is not contractible, reputational concerns may prevent the platform provider from cheating sellers. That \( p_0 \) is known by sellers when they have to decide whether to accept contracts is standard in some industries such as videogames (see Hagiu, 2006, for example). If \( p_0 \) were chosen after sellers have decided whether to accept the platform’s offers, sellers would anticipate a hold-up problem that would backfire and harm the platform.
In the second part of the paper (to be done), we study some variations of the game in Section 5. First, we assume that the platform provider can commit to serving only one seller, and study under which conditions it may choose to do so. Second, we allow for \( n \geq 2 \) and free entry of sellers.

In Sections 3 and 4, we seek for symmetric subgame perfect equilibria (SPE). In Section 5, we seek for symmetric Perfect Bayesian Equilibria (PBE) given standard constraints on how off-the-equilibrium-path beliefs are formed.

### 3. Public contracts in a one-sided market

We start by studying the second stage (recall that in this section and the following two, we assume \( n = 2 \)). After observing \( p_i \) (\( i = 1, 2 \)), consumers who have purchased the platform good decide their demands for the sellers’ products. Looking at interior solutions of a consumer’s utility maximization problem yields the following per-capita demand for the product of seller \( i \):

\[
q_i(p_i, p_j) = \frac{1 - \theta - p_i + \theta p_j}{1 - \theta^2}.
\]  

(1)

Per-capita consumption does not depend on the distance between the consumer and the platform. Thus, the overall demand for seller \( i \)’s product is \( Q_i(p_i, p_j) = x_0 q_i(p_i, p_j) \), where \( x_0 \) is the number of consumers who choose to buy the platform good in the first stage. Seller \( i \in \{1, 2\} \) solves the following problem given a price \( p_j \) by the other seller:

\[
\max_{p_i} (p_i - w_i) Q_i(p_i, p_j) - f_i,
\]

where \( f_i \) is a cost already sunk and the total number of consumers, \( x_0 \), is given from the first stage. Seller \( i \)’s first-order condition is

\[
x_0 (1 - \theta - 2p_i + w_i + \theta p_j) = 0,
\]

so its equilibrium price is

\[
p_i(w_i, w_j) = \frac{(2 + \theta)(1 - \theta) + 2w_i + \theta w_j}{(2 + \theta)(2 - \theta)}.
\]  

(2)

It readily follows from (1) that each consumer buys

\[
q_i(w_i, w_j) = \frac{(1 - \theta)(2 + \theta) - w_i(2 - \theta^2) + \theta w_j}{(1 - \theta^2)(4 - \theta^2)}
\]

(3)

units of product \( i \) \( (i, j = 1, 2, i \neq j) \).

We now turn to the analysis of the first stage. By symmetry, optimal royalties are such that \( w_1 = w_2 = w \). Recall that in the one-sided market case, \( p_0 = 0 \). In
the first stage, given $w$, the utility of consumer $x$ is

$$U^o_x(w) = \frac{(1 - w)^2}{(1 + \theta)(2 - \theta)^2} - x,$$

where the superscript $o$ refers to the one-sided, public contracts case. This results in a demand for the platform good equal to

$$x^o_0(w) = \frac{(1 - w)^2}{(1 + \theta)(2 - \theta)^2}.$$

Anticipating how play will evolve in the second stage, seller $i$ will accept the contract offered by the platform if and only if $f_i \leq x^0_0(p_i - w_i)q_i$. Thus, the platform provider sets $f_i = x^0_0(p_i - w_i)q_i$, and solves

$$\max_w x^o_0(w) \left[ \sum_{i=1}^{2} p_i(w, w)q_i(w, w) \right]. \quad (4)$$

It is then easy to prove the following result.

**Proposition 1.** If firm 0 provides a one-sided platform and contracts are publicly observed by all parties, then the equilibrium royalties equal

$$w^o = \frac{3\theta - 2}{4},$$

so the equilibrium price charged by seller $i \in \{1, 2\}$ is

$$p^o_i = \frac{1}{4},$$

per-capita consumption of each product is

$$q^o_i = \frac{3}{4(1 + \theta)},$$

and the number of consumers is

$$x^o_0 = \frac{9}{16(1 + \theta)}.$$

Finally, platform profits are

$$\pi^o_0 = \frac{27}{128(1 + \theta)^2},$$

and consumer surplus is

$$cs^o = \frac{81}{512(1 + \theta)^2}.$$

Note that $w^o < 0$ for $\theta < 2/3$ and $w^o > 0$ for $\theta > 2/3$. To understand this result, note on the one hand that, given that the platform perfectly predicts second stage prices as a function of royalties, it can solve the problem in expression (4) as if
it was choosing prices $p_i$ instead of royalties $w_i$. The first-order condition with respect to price $p_1$ is:

$$x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} + p_2 \frac{\partial q_2}{\partial p_1} \right) + \frac{\partial x_0}{\partial p_1} (p_1 q_1 + p_2 q_2) = 0.$$ 

Seller 1, on the other hand, chooses price $p_1$ according to the following first-order condition:

$$x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} - w_1 \frac{\partial q_1}{\partial p_1} \right) = 0.$$ 

When choosing a price, seller 1 ignores two effects: the effect a change in $p_1$ has on the per-capita demand of seller 2, and the effect it has on the number of consumers who join the platform. Note that the first effect is positive or negative depending on whether $\partial q_2 / \partial p_1$ is positive or negative, and the second effect is always negative.

It is straightforward to see that the platform can make seller 1 internalize these two effects by choosing an appropriate royalty fee. In particular, it should choose a royalty fee so that

$$-w_1 \frac{\partial q_1}{\partial p_1} = p_2 \frac{\partial q_2}{\partial p_1} + \frac{\partial x_0}{\partial p_1} \frac{p_1 q_1 + p_2 q_2}{x_0}.$$ 

When $\theta \leq 0$, the two terms on the right hand side are negative. Thus, the optimal royalty fee is negative. The royalty fee will be positive only if $\theta$ is positive and sufficiently large to overcome the negative effect of the change in the number of consumers joining the platform. This is precisely the result in Proposition 1.

### 4. Public Contracts in a Two-Sided Market

We now allow the platform to be priced at $p_0 \neq 0$. Second-stage decisions (for a given number of consumers and pair of royalty fees) are equivalent to those of the previous section (see expressions (2) and (3)). In the first stage, given $w_1 = w_2 = w$ and $p_0$, the utility of consumer $x$ is

$$U^t_x(w, p_0) = \frac{(1 - w)^2}{(1 + \theta)(2 - \theta)^2} - x - p_0,$$

where the superscript $t$ refers to the two-sided, public contracts case. It follows that the demand for the platform good is

$$x^t_0(w, p_0) = \frac{(1 - w)^2}{(1 + \theta)(2 - \theta)^2} - p_0.$$
As in the previous section, the platform provider sets $f_i = x_0 (p_i - w_i) q_i$, but now solves

$$\max_{w, p_0} x_0^t(w, p_0) \left[ p_0 + p_1(w, w) q_1(w, w) + p_2(w, w) q_2(w, w) \right],$$

which leads to the following result.

**Proposition 2.** If firm 0 provides a two-sided platform and contracts are publicly observed by all parties, then the equilibrium royalties equal

$$w^t = -(1 - \theta) < 0,$$

and the equilibrium price for the platform equals

$$p_0^t = \frac{1}{2(1 + \theta)} > 0.$$

The equilibrium price charged by seller $i \in \{1, 2\}$ is

$$p_i^t = 0,$$

per-capita consumption of each product is

$$q_i^t = \frac{1}{1 + \theta},$$

and the number of consumers is

$$x_0^t = \frac{1}{2(1 + \theta)}.$$

Finally, platform profits are

$$\pi_0^t = \frac{1}{4(1 + \theta)^2},$$

and consumer surplus is

$$cs^t = \frac{1}{8(1 + \theta)^2}.$$

Note that, in contrast with the previous case, in this case the optimal royalty fee is always negative, and goes to zero as $\theta \to 1$. To understand this result, we can proceed in a similar way as before. We start by noting that the first-order condition of the platform with respect to price $p_0$ is:

$$x_0 + \frac{\partial x_0}{\partial p_0} (p_0 + p_1 q_1 + p_2 q_2) = 0.$$

If the platform acts as if it was choosing price $p_1$ instead of royalty fee $w_1$, it would choose price $p_1$ according to the following first-order condition:

$$x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} + p_2 \frac{\partial q_2}{\partial p_1} \right) + \frac{\partial x_0}{\partial p_1} (p_0 + p_1 q_1 + p_2 q_2) = 0.$$
The first-order condition with respect to $p_0$ implies that

$$p_0 + p_1 q_1 + p_2 q_2 = x_0,$$

given that $\partial x_0 / \partial p_0 = -1$. Since Roy’s identity implies that $\partial x_0 / \partial p_1 = -q_1$, the first-order condition becomes:

$$x_0 \left( p_1 \frac{\partial q_1}{\partial p_1} + p_2 \frac{\partial q_2}{\partial p_1} \right) = 0.$$

In a symmetric equilibrium:

$$x_0 \cdot p_1 \left( \frac{\partial q_1}{\partial p_1} + \frac{\partial q_2}{\partial p_1} \right) = 0,$$

so it is optimal to set a royalty that induce sellers to sell their products at a price of zero. Given that seller 1 chooses price $p_1$ so that

$$x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} - w_1 \frac{\partial q_1}{\partial p_1} \right) = 0,$$

the royalty fee must be negative so that sellers choose prices equal to zero. Finally, note that prices go to marginal cost as $\theta \to 1$ due to pure Bertrand competition, so the royalty converges to zero as products become perfect substitutes. Thus the results in Proposition 2.

5. Private contracts

In this section, we assume contracts between the platform and the sellers are private. Thus, consumers cannot observe any of the contracts offered to sellers, and a seller can only observe the contract it is offered. We will seek for symmetric Perfect Bayesian Equilibria (PBE) given standard constraints on how off-the-equilibrium-path beliefs are formed.\textsuperscript{6} In what follows, let $p_0^*$ denote the price charged to consumers by the platform provider in a symmetric PBE. Also, let $w^*$ denote the royalty fee that is offered to seller $i \in \{1, 2\}$ in a symmetric PBE, and $f^*$ the associated fixed fee.

Regarding the formation of out-of-equilibrium beliefs, note that, upon observing any $p_0 \neq p_0^*$, rational consumers would realize that such a deviation affects sellers’ profits and potentially their incentives to enter the market (this happens when $p_0 > p_0^*$). They should therefore conclude that a price deviation must be accompanied by a change in the fixed fee and/or a change in the royalty fee offered to each seller. We will look at equilibria in which consumers rationalize any price

\textsuperscript{6}No asymmetric equilibrium exists, so the symmetry requirement is without loss of generality, at least if one restricts attention to equilibria in which the pricing strategy and beliefs held by a seller are polynomial functions of the royalties it observes.
deviation by conjecturing that there was no deviation in the royalty fee offered to each seller; hence, consumers believe upon observing \( p_0 \neq p_0^* \) that the platform is simply adjusting the fixed fee offered to each seller just to make it break-even given \( w^* \). These beliefs are in the spirit of “passive beliefs” (Hart and Tirole, 1990), but they require some rationability by consumers. In particular, when consumers observe a price deviation, they acknowledge that this should have had an impact on the sellers’ willingness to accept the contract, and they reason that the absence of such an impact must be due to a change in the fixed fee offered to each seller. We refer to this weak form of passive beliefs held by consumers as “weakly passive beliefs.”\(^7\) Note that the main implication of such belief formation is that consumers always expect the interaction of sellers in the product market to be unaffected by the choice of \( p_0 \).

Because a seller anticipates such unsophisticated behavior by consumers when \( p_0 \neq p_0^* \), it believes that \( p_0 \neq p_0^* \) conveys no information about contract offers. Thus, sellers therefore form passive beliefs with respect to deviations in consumer prices by firm 0. However, seller \( i \in \{1, 2\} \) is assumed to form “wary beliefs” (McAfee and Schwartz, 1994; Rey and Vergé, 2004) when it observes an unexpected contract offer. In such cases, it believes that the platform provider must have made an offer to \( j \in \{1, 2\} \) \((j \neq i)\) that maximizes the platform’s total profit given the price that it charges to consumers and the contract offered to seller \( i \in \{1, 2\} \). Of course, in equilibrium, a seller estimates perfectly the offer made by the platform to the other seller, but the formation of wary beliefs by sellers implies that, if the platform deviates from equilibrium play, then sellers will correctly infer how it is deviating. We also assume that a seller that forms wary beliefs conjectures that the other seller also does, and also conjectures that the platform provider does not want to drive any seller out of the market.

5.1. Contracts observable to sellers, but unobservable to consumers.

Before examining equilibrium play when the contract offer received by a seller is solely observed by such a seller, it is useful to examine the benchmark case in which sellers observe each other’s contract, but consumers do not. Let us denote

\(^7\)The outcome would be the same under the standard strong form of passive beliefs (corresponding to situations in which consumers do not change their equilibrium beliefs when observing out-of-equilibrium behavior). However, it would be harder to interpret some situations. For example, upon observing \( p_0 > p_0^* \), a consumer who kept her beliefs about \( f^* \) and \( w^* \) should conclude that the sellers are accepting a contract that makes them lose money, for consumer demand is smaller than it should be in equilibrium (since we shall show later on that consumer demand for the platform does not affect competition between sellers, which solely depends on royalty fees).
the contract offered to each seller in equilibrium by \((\hat{f}, \hat{w})\). Because consumers cannot observe deviations from this contract and form weakly passive beliefs when observing any \(p_0\), their demand for the platform when observing price \(p_0\) equals
\[
x_0(p_0, \hat{w}) = \frac{(1 - \hat{w})^2}{(1 + \theta)(2 - \theta)^2} - p_0.
\]

Taking into account that the platform provider extracts all the surplus from the sellers, it follows that it chooses \(p_0\), \(w_1\) and \(w_2\) to maximize
\[
x_0(p_0, \hat{w})[p_0 + p_1(w_1, w_2)q_1(w_1, w_2) + p_2(w_2, w_1)q_2(w_2, w_1)].
\]

The first-order condition corresponding to \(w_i\) is as follows:
\[
\theta(1 - \theta)(2 + \theta)^2 - (8 - 6\theta^2)w_i + 2\theta^3w_j = 0 \quad (i, j = 1, 2; \; i \neq j).
\]
Rearranging this equation allows us to give it an interpretation that will be useful later on: when seller \(i\) receives an offer involving royalty fee \(w\) \((i \in \{1, 2\})\), it infers that the platform provider finds it optimal to charge seller \(j \in \{1, 2\} \; (j \neq i)\) with a royalty fee equal to
\[
\hat{w}^*(w) = \frac{\theta(1 - \theta)(2 + \theta)^2 + 2\theta^3w}{2(4 - 3\theta^2)}.
\]

As a result, \(\hat{w}^*(\cdot)\) can be interpreted as a seller’s belief about the royalty fee offered to the other seller. Such a belief is correct both on and off the equilibrium path because the platform anticipates that sellers will have complete information when pricing, so there is no way to fool them. The function \(\hat{w}^*(\cdot)\) will serve as a useful benchmark when we further assume in the next subsection that sellers cannot observe each other’s contract offers.

To fully solve the model, it is easy to show that in equilibrium it must hold that
\[
\hat{w} = \frac{\theta}{2}.
\]
Thus, the royalty fee is positive if \(\theta > 0\) and negative if \(\theta < 0\). The first-order condition corresponding to \(p_0\) can be written as
\[
\frac{(2 - \theta)^2}{4(1 + \theta)(2 - \theta)^2} - \frac{1}{2(1 + \theta)} - 2p_0 = 0,
\]
so
\[
\hat{p}_0 = -\frac{1}{8(1 + \theta)} < 0.
\]
In equilibrium, the platform induces seller $i$ to charge price
\[ \hat{p}_i = \frac{1}{2} > 0 \]
and gains
\[ \hat{\pi}_0 = \frac{9}{64 (1 + \theta)^2}. \]

To understand these results, we can proceed as in the previous sections. If the platform acts as if it was choosing price $p_1$ instead of royalty fee $w_1$, it would choose price $p_1$ according to the following first order condition:
\[ x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} + p_2 \frac{\partial q_2}{\partial p_1} \right) = 0. \]
Note that this first-order condition differs from that of the previous case because consumers do not observe changes in royalty fees, so their decision to buy the platform good depends only on their beliefs about the equilibrium royalty. In a symmetric equilibrium, it holds that
\[ -\left( \frac{\partial q_1}{\partial p_1} + \frac{\partial q_2}{\partial p_1} \right) p_i = q_i. \]
Thus, the optimal implied price for sellers is positive. This contrasts with the result in the public contracts case, in which the optimal price was zero.

It is easy to see that the optimal price $p_0$ solves
\[ p_0 = \frac{U(p_1, p_2) - p_1 q_1 - p_2 q_2}{2}. \]
This equation shows that the platform has incentives to lower $p_0$, in comparison with the public contracts case, for two reasons: to compensate the decrease in consumer surplus from consumption of seller goods ($U(p_1, p_2) < U(0, 0)$), and because seller surplus per consumer increases ($p_1 q_1 + p_2 q_2 > 0$). In the case at hand, it turns out that the platform finds it optimal to set a negative access fee for consumers.

Finally, note that seller $i$ chooses price $p_i$ so that
\[ x_0 \left( q_1 + p_1 \frac{\partial q_1}{\partial p_1} - w_1 \frac{\partial q_1}{\partial p_1} \right) = 0. \]
Thus, the royalty needs to be positive if the cross-price effect $\frac{\partial q_2}{\partial p_1}$ is positive, and negative if the cross-price effect is negative.

Summarizing, we find that when consumers do not observe royalty fees, they are less reactive to changes in the intensity of competition between sellers, since they cannot observe deviations from the royalty fees they expect in equilibrium.
As a consequence, the platform has incentives to behave opportunistically, and choose royalties to induce collusive pricing by sellers. This is correctly anticipated by consumers, and their utility from having access to the platform decreases. The platform has incentives to lower access prices for consumers for two reasons: to compensate the lower demand for platform access, and because seller revenue per consumer increases.

5.2. Contracts unobservable to sellers and consumers. We now turn to the analysis of the cases in which the contract offer received by a seller is solely observed by such a seller, starting with the second stage.

At the beginning of the second stage, seller $i \in \{1, 2\}$ knows $p_0$, $x_0$, $f_i$ and $w_i$, and has to choose a price for its product based on this information. Taking into account that seller $i$’s overall demand product equals $Q_i(p_i, p_j) \equiv x_0q_i(p_i, p_j)$, we can solve for the second-stage subgames. Recalling that we are examining symmetric equilibria, let $B(\hat{w})$ denote the belief formed by seller $i \in \{1, 2\}$ about the royalty fee paid by seller $j \in \{1, 2\} (j \neq i)$ to the platform provider.\footnote{Because we are looking at symmetric equilibria, the belief function $B(\cdot)$ does not depend on the label of the seller receiving the unexpected offer. Note that, in general, $B(\cdot)$ is an unrestricted function except for the constraint that $B(w^*) = w^*$ (i.e., conjectured beliefs are fulfilled along the equilibrium path). In our case, we will restrict the function so that beliefs are wary.} We follow Rey and Vergé (2004), and restrict attention to equilibria in which seller $i$’s belief about the royalty fee paid by the other seller does not depend on the fixed fee it observes. Not only is the pricing strategy of seller $i \in \{1, 2\}$ independent from the fixed fee it already paid, but it is also independent from $p_0$ (and hence from $x_0$). Such a price has no signaling role and it does not affect belief formation, which seems a reasonable assumption given that $x_0$ is simply a scaling factor in seller $i$’s second-stage profit.\footnote{Therefore, it does not affect equilibrium pricing in the second-second if sellers believe that it does not convey some information, making it self-fulfilling that it is pointless for the platform provider to use it for signaling purposes.}

In what follows, let $p_i(w_i)$ denote the strategy of seller $i \in \{1, 2\}$ in the second-stage subgame if it has observed an offer of $(w_i, f_i)$ and price $p_0$. Having observed this, seller $i \in \{1, 2\}$ chooses $p_i$ to maximize $(p_i - w_i)Q_i(p_i, p_j(B(w_i))) - f_i (j \in \{1, 2\}; j \neq i)$ with $f_i$ already sunk, so its first-order condition is

$$1 - \theta + w_i - 2p_i(w_i) + \theta p_j(B(w_i)) = 0. \quad (6)$$

We now turn to analyzing the first stage of play. Regardless of the price $p_0$ that consumers observe, they believe that seller $i \in \{1, 2\}$ is charged a royalty fee of
$w^*$, so they expect a price

$$p_i^* = \frac{1 - \theta + w^*}{2 - \theta}$$

for each unit they purchase from seller $i \in \{1, 2\}$ in the second stage. Given price $p_0$, the overall utility expected by consumer $x$ equals

$$U_x(w^*, p_0) = \frac{(1-w^*)^2}{(1+\theta)(2-\theta)^2} - x - p_0,$$

so the demand for the platform good is

$$x(w^*, p_0) = \frac{(1-w^*)^2}{(1+\theta)(2-\theta)^2} - p_0.$$

The platform provider’s total profit if it charges $p_0$ and makes a private offer of $(w_1, f_1)$ and $(w_2, f_2)$ to sellers 1 and 2, respectively, is as follows:

$$
\pi_0(w_1, f_1, w_2, f_2, p_0) = x(p_0) \left[ p_0 + w_1 q_1(p_1(w_1), p_2(w_2)) + w_2 q_2(p_1(w_1), p_2(w_2)) \right] + f_1 + f_2,
$$

since the platform provider can perfectly forecast actual sales made by sellers 1 and 2. In order for seller 2 (say) to form wary beliefs, the inference made by such a seller about seller 1’s contract upon observing a price of $p_0$ and an offer of $(w_2, f_2)$ must be such that $B(w_2)$ maximizes $\pi_0(w, f, w_2, f_2, p_0)$ with respect to $w$ and $f$ subject to the constraint that $f \leq (p_1(w) - w)x(p_0)q_1(p_1(w), p_2(B(w)))$.

Taking into account that the constraint must bind at the optimum and that

$$q_1(p_1(w), p_2(B(w))) = \frac{p_1(w) - w}{1 - \theta^2},$$

by condition (6) yields that

$$B(w_2) \in \arg\max_w \pi_0(w, w_2, f_2, p_0),$$

where

$$
\pi_0(w, w_2, f_2, p_0) = x(p_0) \left\{ p_0 + w q_1(p_1(w), p_2(w_2)) + w_2 q_2(p_2(w_2), p_1(w)) + \frac{[p_1(w) - w]^2}{1 - \theta^2} \right\} + f_2.
$$
Maximizing $\pi_0(w, w_2, f_2, p_0)$ with respect to $w$ yields the following first-order condition:

$$
0 = q_1(p_1(w), p_2(w_2)) + \frac{2[p_1(w) - w]}{1 - \theta^2} \left( \frac{dp_1(w)}{dw} - 1 \right) + \left[ w \frac{\partial q_1(p_1(w), p_2(w_2))}{\partial p_1} + w_2 \frac{\partial q_2(p_2(w_2), p_1(w))}{\partial p_1} \right] \frac{dp_1(w)}{dw}.
$$

(8)

Since our purpose at this stage is to build some intuition, let us assume for now that a unique solution to equation (8) exists for any $w_2$, denote it by $w^*_1(w_2)$, and note that it must coincide with $B(w_2)$ even if $w_2 \neq w^*$ because sellers form wary beliefs even when off the equilibrium path. Using the implicit function theorem, we obtain the following result:

$$
\frac{dB(w_2)}{dw_2} = \frac{dw^*_1(w_2)}{dw_2} = -\frac{\theta}{1 - \theta^2} \left( \frac{dp_2(w_2)}{dw_2} + \frac{dp_1(w)}{dw} \right) + \partial_2 \pi_0(w, w_2, f_2, p_0) / \partial w_2.
$$

If $\pi_0(w, w_2, f_2, p_0)$ is strictly concave with respect to $w$ (as we shall later show), symmetry yields that

$$
\operatorname{sign} \left( \frac{dB(w)}{dw} \right) = \operatorname{sign} \left( \theta \frac{dp(w)}{dw} \right).
$$

Whenever it holds that $dp(w)/dw > 0$, which is an intuitive property that equilibrium prices should satisfy,\(^{10}\) we have that $dB(w)/dw \geq 0$ if and only if $\theta \geq 0$, according well with what one may have expected. Sellers’ prices are strategic complements if $\theta > 0$ and strategic substitutes otherwise (provided goods are not independent), and the platform provider aims at softening competition between sellers under strategic complementarity and at toughening such competition under strategic substitutability.

Having shed some light on some of the properties that the equilibrium satisfies, we proceed to showing existence and characterizing it. To this end, evaluating the first-order condition at $w = B(w_2)$ (recall condition (7)) and letting $p_i(w) = p(w)$ because of symmetry yields that the following equation must hold:

$$
0 = 1 - \theta - p(B(w_2)) + \theta p(w_2) + (\theta w_2 - B(w_2)) \frac{dp(B(w_2))}{dw} + 2[p(B(w_2)) - B(w_2)] \left[ \frac{dp(B(w_2))}{dw} - 1 \right].
$$

(9)

\(^{10}\)Note that we shall restrict attention to polynomial pricing strategies, and that in such cases there is no loss in further restricting them to be affine.
If one focuses on PBE such that \( p(\cdot) \) and \( B(\cdot) \) be polynomial functions, then Rey and Vergé (2004) show that there is no loss of generality in restricting attention to affine functions, so one can readily solve the system of differential equations given by (9) and (6) (after dropping subscripts) to obtain the following result.

**Proposition 3.** The unique symmetric PBE in which \( p(w) \) and \( B(w) \) are polynomial functions is such that \( p(w) = \Theta_\theta + \Sigma_\theta w \) and \( B(w) = \Gamma_\theta + \Phi_\theta w \) for some constants \( \Theta_\theta \in [0, 1], \Sigma_\theta \in [\frac{1}{2}, 1] > 0, \Gamma_\theta \in [0, 1], \) and \( \Phi_\theta \in [-1, 1] \). In such an equilibrium, it always holds that

\[
p_i^* = \Theta_\theta + \frac{\Sigma_\theta \Gamma_\theta}{1 - \Phi_\theta} \geq 0 \quad (i = 1, 2)
\]

and \( w^* \geq 0 \) for any \( \theta \in (-1, 1) \), with \( w^* = 0 \) if and only if \( \theta = 0 \). Also, platform profits are

\[
\pi_0^* = \left( \frac{1 - (p_i^*)^2}{2(1 + \theta)} \right)^2,
\]

and consumer surplus is

\[
cs^* = \frac{1}{2} \left( \frac{1 - (p_i^*)^2}{2(1 + \theta)} \right)^2.
\]

**Proof.** See Appendix A. □

Contrary to the case in which sellers can observe each other’s royalties (sub-section 5.1), royalty fees are never negative under private contracting, regardless of whether price competition between sellers displays strategic complementarity (\( \theta > 0 \)) or strategic substitutability (\( \theta < 0 \)). When sellers can observe each other’s royalties, \( \theta > 0 \) implies that \( d\hat{w}^*(w)/dw > 0 \) (see expression (5)), so an increase in the royalty fee a seller observed would (correctly) make it believe that the other seller’s royalty offer must have increased, since the platform aims at softening competition, and hence in equilibrium \( \hat{w} = \theta/2 > 0 \); the converse happens if \( \theta < 0 \) (so that \( d\hat{w}^*(w)/dw < 0 \)), with \( \hat{w} = \theta/2 < 0 \) in these cases because the platform wishes to toughen competition. When sellers cannot observe each other’s offers, their beliefs become more sensitive to observed royalties. This overreaction to changes in the royalty fee observed is a straightforward effect of the wary beliefs formed by sellers in face of opportunistic contracting by the platform. Figure 1 plots \( d\hat{w}^*(w)/dw \) (see solid curve) relative to \( dB(w)/dw \) (see dashed curve) as parameter \( \theta \) varies.

The determinants of how the equilibrium royalty fee relates to \( \theta \) are different when sellers can observe each other’s royalty offers and when they cannot. When they can observe them as in subsection 5.1, the platform’s incentives to deviate
have to do with making competition between sellers softer (if $\theta > 0$) or tougher (if $\theta < 0$), as we just mentioned. When sellers cannot observe each other’s royalty offers, the platform’s incentives to deviate greatly depend on how a seller that receives an unexpected offer believes it is being treated. In particular, such a seller (correctly) infers that the platform must be simultaneously deviating with the other seller in a way that the opportunistic platform does not care about seller 2’s profitability. Indeed, taking into account that the platform extracts all the surplus that seller $i$ expects to make when observing royalty fee $w_i$, it holds that the payoff to the platform if it chooses $w_1, w_2$ and $p_0$ equals

\[
\hat{\pi}_0(w_1, w_2, p_0) = x(p_0) \left\{ p_0 + w_1 q_1(p_1(w_1), p_2(w_2)) + w_2 q_2(p_2(w_2), p_1(w_1)) \\
+ [p_1(w_1) - w_1] q_1(p_1(w_1), p_2(B(w_1))) \\
+ [p_2(w_2) - w_2] q_2(p_2(w_2), p_1(B(w_2))) \right\}
\]

\[
= x(p_0) \left\{ p_0 + w_1 q_1(p_1(w_1), p_2(w_2)) + w_2 q_2(p_2(w_2), p_1(w_1)) \\
+ [p_1(w_1) - w_1] q_1(p_1(w_1), p_2(w_2)) \\
+ [p_1(w_1) - w_1] [q_1(p_1(w_1), p_2(B(w_1))) - q_1(p_1(w_1), p_2(w_2))] \\
+ [p_2(w_2) - w_2] q_2(p_2(w_2), p_1(B(w_2))) \right\}.
\]

Clearly, maximizing this payoff with respect to $w_1$ is equivalent to maximizing

\[
\hat{\pi}'_0(w_1, w_2) = [p_1(w_1) - w_1] q_1(p_1(w_1), p_2(w_2)) \\
+ w_1 q_1(p_1(w_1), p_2(w_2)) + w_2 q_2(p_2(w_2), p_1(w_1)) \\
+ [p_1(w_1) - w_1] [q_1(p_1(w_1), p_2(B(w_1))) - q_1(p_1(w_1), p_2(w_2))].
\]
so the platform cares about seller 1’s actual profit, the actual royalty revenue generated by each seller and the change in seller 1’s profit because of the formation of wary beliefs. By the envelope theorem, seller 1’s actual profit when \( w_1 \) varies a bit is equal to 

\[-q_1(p_1(w_1),p_2(w_2)), \text{ so } w^*_1 = B(w^*_1) \text{ implies that}\]

\[
\frac{\partial \hat{\pi}'_0(w_1,w_2)}{\partial w_1}\bigg|_{w_1=w_2=w^*_1} = 0
\]

is equivalent to

\[
\left\{ [p(w^*_1) - w^*_1] \frac{\partial B(w^*_1)}{\partial w} - (1 - \theta) w^*_1 \right\} \frac{dp(w^*_1)}{dw} = 0.
\]

The fact that \( p(w^*_1) > w^*_1 \) then implies that

\[
w^*_1 = \frac{\theta}{1 - \theta} \frac{dB(w^*_1)}{dw} [p(w^*_1) - w^*_1]
\]

must be nonnegative because we showed earlier that \( \theta(dB(w^*_1)/dw) \geq 0. \)

As we have shown, the sign of \( w^*_1 \) depends on how the second argument of \( q_1(p_1(w_1),p_2(B(w_1))) \) varies with \( w_1 \), that is, on whether an increase in \( w_1 \) will stimulate seller 1’s sales via the conjectured price change performed by seller 2. Because seller 1 always believes that this is indeed the case, \( w^*_1 \) is always nonnegative. When sellers can observe each other’s offer, we showed in subsection 5.1 that the equilibrium royalty fee is positive if and only if competition between sellers displays strategic complementarity. Figure 2 compares royalty fees in the two models (the dashed curve corresponds to the case of private contracting).

![Figure 2. Comparison of royalty fees](image)

Because \( w^*_1 < \hat{w} \) if and only if \( \theta > 0 \), it should come as no surprise that the comparison of sellers’ prices in both situations is as illustrated by Figure 3 (the
dashed curve represents the situation when seller cannot observe each other’s offer).

![Figure 3. Comparison of seller prices](image)

Relative to when sellers can observe each other’s contract, it holds when they cannot that the platform provider loses part of its market power vis-à-vis sellers because of its opportunistic behavior when dealing with each on a one-on-one basis (as in Rey and Vergé, 2004). This smaller market power implies that the platform provider cannot sufficiently raise sellers’ prices through the royalty fees when goods are substitutes; when goods are complements, the smaller market power of the platform provider implies that it cannot sufficiently lower prices charged by sellers so as to mitigate the double marginalization problem first pointed out by Cournot (1838) for the case of perfect complements.

The difference in pricing by sellers illustrated by the previous figure has key implications for platform pricing, since one of the two determinants of platform demand is how much utility consumers expect to attain given the anticipated pricing by sellers. When $\theta < 0$, consumers correctly anticipate that sellers will charge higher prices when they cannot observe each other’s offer than when they can, so the platform provider has an incentive to lower the platform’s price relative to when sellers can observe each other’s offer. When $\theta > 0$, the sellers charge lower prices when they cannot observe each other’s offer than when they can, so the platform provider has an incentive to raise the platform’s price relative to when sellers can observe each other’s offer.

The other determinant of platform pricing is how much overall profit is generated per consumer through the two sellers. Figure 4 shows how total profit generated by sellers per customer varies with $\theta$ (the dashed curve represents the situation when seller cannot observe each other’s offer).
Because sellers are induced to price collusively when they can observe each other’s offer, it holds that per-consumer profitability is at least as large as when they cannot observe each other’s offer. This implies that, regardless of the value of $\theta$, the platform provider has an incentive to set a higher price for the platform when sellers cannot observe each other’s offer than when they can. Interestingly, note that the incentive is very small when $\theta > 0$: in such cases, the platform provider’s opportunistic behavior is hardly costly in terms of generating sellers’ profits. The effect highlighted by Rey and Vergé (2004) is really mild.

Overall, we find that pricing by the platform is driven by the anticipated effect of sellers’ prices on consumer utility. On the one hand, when $\theta > 0$, the platform provider prices higher when sellers cannot observe each other’s offer than when they can: the effect on consumer demand of having lower prices dominates the effect of appropriating less profit through sellers. On the other hand, when $\theta < 0$, the effect of having lower consumer utility when sellers cannot observe each other’s offer always dominates the lower per-consumer profitability that arises when sellers cannot observe each other’s offer. This is illustrated by Figure 5 (the dashed curve represents the situation when seller cannot observe each other’s offer).

It should then not be very surprising that platform profits are greater when sellers cannot observe each other’s offer than when they can if and only if $\theta > 0$, as the Figure 6 shows (the dashed curve represents the situation when seller cannot observe each other’s offer).

A similar result holds for consumer and total welfare, since they are proportional to platform profits both when sellers cannot observe each other’s offer and when they can.
We now turn to our main result. In particular, the following proposition shows the effects of private contracts in a two-sided market by comparing the equilibrium of this subsection with the equilibria of the previous two sections.

**Proposition 4.** Equilibrium royalties can be positive or negative in a one-sided market with public contracts, are negative in a two-sided market with public contracts, and are positive in a two-sided market with private contracts. The price of the platform good for consumers is positive in a two-sided market with public contracts, and is negative in a two-sided market with private contracts. Comparing two-sided market models, private contracts lead to lower profit, consumer surplus, and welfare.

The first two claims in the proposition follow from comparing the equilibria of the models in Sections 3, 4, and 5, and the proof for the last claim is in the proof of Proposition 3.
Proof of Proposition 3. If $p(w) = \Theta + \Sigma w$ and $B(w) = \Gamma + \Phi w$ for some parameters $\Theta$, $\Sigma$, $\Gamma$ and $\Phi$ to be determined, conditions (6) and (9) can be rewritten as

$$(1-\theta)(1-\Theta) - 2\Sigma \Gamma + (\Theta + \Sigma \Gamma - \Gamma)2(\Sigma - 1) + [2\Sigma(\theta - \Phi) + \Phi(\Sigma - 1)2(\Sigma - 1)]w_2 = 0$$

and

$$1 - \theta + \theta \Sigma \Gamma - (2 - \theta)\Theta + (1 - 2\Sigma + \theta \Sigma \Phi)w_2 = 0.$$ 

Since these two conditions should be satisfied for all $w_2$, we must have

$$(1-\theta)(1-\Theta) - 2\Sigma \Gamma + (\Theta + \Sigma \Gamma - \Gamma)2(\Sigma - 1) = 0,$$  \hspace{1cm} (10)

$$2\Sigma(\theta - \Phi) + \Phi(\Sigma - 1)2(\Sigma - 1) = 0,$$  \hspace{1cm} (11)

$$1 - \theta + \theta \Sigma \Gamma - (2 - \theta)\Theta = 0$$  \hspace{1cm} (12)

and

$$1 - 2\Sigma + \theta \Sigma \Phi = 0.$$  \hspace{1cm} (13)

Rey and Vergé (2004) have already shown that there exists a unique tuple $(\Theta, \Sigma, \Gamma, \Phi)$ that solves these equations and the required second-order conditions for the platform's maximization program, but we will give closed-form solutions that will prove useful later on.

When $\theta = 0$, it is easy to see that there is a unique solution to equations (10)-(13), given by $\Theta = 1/2$, $\Sigma = 1/2$, $\Gamma = 0$ and $\Phi = 0$. From (13), one obtains

$$\Phi = \frac{2\Sigma - 1}{\theta \Sigma},$$

since it can be shown that there can be no solution with $\Sigma = 0$. Plugging this value for $\Phi$ in (11) allows us to rewrite it as the following cubic equation:

$$\Sigma^3 - \left(\frac{7 - \theta^2}{2}\right) \Sigma^2 + \frac{5}{2} \Sigma - \frac{1}{2} = 0.$$  \hspace{1cm} (14)

Letting

$$a \equiv -\frac{7 - \theta^2}{2},$$

$$b \equiv \frac{5}{2},$$

$$c \equiv -\frac{1}{2},$$

$$K \equiv \frac{3b - a^2}{9}$$
and
\[ L \equiv \frac{9ab - 27c - 2a^3}{54}, \]
the solutions to the cubic equation are the following:
\[ \Sigma_k = 2\sqrt{-K} \cos \left( \frac{1}{3} \arccos \left( \frac{L}{\sqrt{-K^3}} \right) + \frac{2\pi k}{3} \right) - \frac{a}{3} \quad (k = 0, 1, 2). \]

The three roots are real, given that the discriminant \( K^3 + L^2 \) is negative for all \( \theta \in (-1, 1) \). Plotting the three roots for all values of \( \theta \), it is easy to see that the only one which is equal to 1/2 when \( \theta = 0 \) is \( \Sigma_2 \). Given that the solution must be continuous in \( \theta \), we know that \( \Sigma = \Sigma_2 \), that is,
\[ \Sigma = \frac{7 - \theta^2}{6} - \frac{(19 - 14\theta^2 + \theta^4)^{1/2}}{3} \sin \left( \frac{\pi}{6} - \frac{1}{3} \arccos \left( \frac{(1 - \theta^2)(82 - 20\theta^2 + \theta^4)}{(19 - 14\theta^2 + \theta^4)^{3/2}} \right) \right). \]

From equation (12), we obtain
\[ \Gamma = \frac{(2 - \theta)\Theta - (1 - \theta)}{\theta \Sigma}, \]
so plugging it into (10) and rearranging yields that
\[ \Theta = \frac{(1 - \theta)((6 + \theta)\Sigma - 2(1 + \Sigma^2))}{4(3 - \Sigma)\Sigma + 2\theta - (3\theta + \theta^2)\Sigma - 4}. \]
It therefore follows from (10) that
\[ \Gamma = \frac{(1 - \theta)(2\Sigma - 1)}{4(3 - \Sigma)\Sigma + 2\theta - (3\theta + \theta^2)\Sigma - 4}. \]

Making it explicit that \( \Theta, \Sigma, \Gamma \) and \( \Phi \) depend on \( \theta \) by writing \( \Theta_\theta, \Sigma_\theta, \Gamma_\theta \) and \( \Phi_\theta \), it is easy to plot them and see that \( 0 \leq \Theta_\theta \leq 1, 1/2 \leq \Sigma_\theta \leq 1, 0 \leq \Gamma_\theta \leq 1 \) and \( -1 \leq \Phi_\theta \leq 1 \) for all \( \theta \in (-1, 1) \). Note that beliefs must be fulfilled in equilibrium, so \( w^* = B(w^*) \) implies that
\[ w^* = \frac{\Gamma_\theta}{1 - \Phi_\theta} \geq 0. \]

Also, the platform should find it optimal to choose \( p_0 = p_0^* \) and \( w_1 = w_2 = w^* \), so \( (w^*, w^*, p_0^*) \in \arg\max_{w_1,w_2,p_0} \hat{\pi}_0(w_1,w_2,p_0) \), where
\[ \hat{\pi}_0(w_1,w_2,p_0) = x(p_0) \left\{ p_0 + w_1 q_1(p_1(w_1), p_2(w_2)) + w_2 q_2(p_2(w_2), p_1(w_1)) + [p_1(w_1) - w_1] q_1(p_1(w_1), p_2(B(w_1))) + [p_2(w_2) - w_2] q_2(p_2(w_2), p_1(B(w_2))) \right\}. \]

Note that the optimal choices of \( w_1 \) and \( w_2 \) do not depend on the choice of \( p_0 \), so the platform provider can maximize with respect to \( w_1 \) and \( w_2 \) ignoring the
value of \( p_0 \); the analysis above leading to expression (9) shows that private offers are chosen optimally, since second-order conditions are satisfied. To see this, note that (8) and the fact that

\[
\frac{dq_1(p_1(w_1), p_2(B(w_1)))}{dw_1} = \frac{1}{1 - \theta^2} \left( \frac{dp_1(w_1)}{dw_1} - 1 \right),
\]

imply that

\[
\frac{\partial^2 \pi_0(w_1, f_1, w_2, f_2, p_0)}{\partial w_1^2} = \frac{2(\Sigma_\theta - \Sigma_\theta + 1)}{1 - \theta^2}
\]

and

\[
\frac{\partial^2 \pi_0(w_1, f_1, w_2, f_2, p_0)}{\partial w_1 \partial w_2} = \frac{2\theta \Sigma_\theta}{1 - \theta^2}.
\]

Thus, it follows from the fact that \( \Sigma_\theta \geq 1/2 \) that

\[
\frac{\partial^2 \pi_0(w_1, f_1, w_2, f_2, p_0)}{\partial w_1^2} \leq 0.
\]

Also, it holds that

\[
\left( \frac{\partial^2 \pi_0(w_1, f_1, w_2, f_2, p_0)}{\partial w_1^2} \right)^2 - \left( \frac{\partial^2 \pi_0(w_1, f_1, w_2, f_2, p_0)}{\partial w_1 \partial w_2} \right)^2 = \Sigma_\theta(\Sigma_\theta - 3\Sigma_\theta + 1)(\Sigma_\theta - 1) - (2\Sigma_\theta - 1)(\Sigma_\theta^2 - 3\Sigma_\theta + 1) - \theta^2 \Sigma_\theta^2 \left( \frac{1 - \theta^2}{2} \right)^2,
\]

which is nonnegative because \( 1/2 \leq \Sigma_\theta \leq 1 \) and \( (2\Sigma_\theta - 1)(\Sigma_\theta^2 - 3\Sigma_\theta + 1) + \theta^2 \Sigma_\theta^2 = 0 \) by (14). Thus, second-order conditions hold.

As for the optimal choice of \( p_0 \) given that seller \( i \in \{1, 2\} \) receives an offer equal to \( (w^*, f^*) \), we need that

\[
0 \leq \frac{dx(p_0)}{dp_0} + \left[ p_0 + 2p_1(w^*)q_1(p_1(w^*), p_2(w^*)) \right] \frac{dx(p_0)}{dp_0} = 0,
\]

so

\[
p_0^* = \frac{(1 - w^*)^2}{2(1 + \theta)(2 - \theta)^2} - \frac{2(\Theta_\theta + \Sigma_\theta w^*)(1 - \Theta_\theta - \Sigma_\theta w^*)}{2(1 + \theta)},
\]

which is negative for all \( \theta \in (-1, 1) \). Finally, note that

\[
p_i^* = \frac{\Sigma_\theta \Gamma_\theta}{1 - \Phi_\theta} (i = 1, 2),
\]

so \( 0 \leq p_i^* \leq 1 \). It readily follows that

\[
q_i^* = \frac{1 - p_i^*}{1 + \theta} > 0,
\]

\[
\pi_0^* = \left( \frac{1 - (p_i^*)^2}{2(1 + \theta)} \right)^2,
\]
and
\[ cs^* = \frac{1}{2} \left( \frac{1 - (p_i^*)^2}{2(1 + \theta)} \right)^2. \]

References


